Comput Optim Appl (2009) 42: 371–392
DOI 10.1007/s10589-007-9122-0

# Benders, metric and cutset inequalities for multicommodity capacitated network design

**Alysson M. Costa · Jean-François Cordeau · Bernard Gendron**

Received: 11 December 2006 / Revised: 9 April 2007 / Published online: 31 October 2007


**Abstract** Solving multicommodity capacitated network design problems is a hard task that requires the use of several strategies like relaxing some constraints and strengthening the model with valid inequalities. In this paper, we compare three sets of inequalities that have been widely used in this context: Benders, metric and cutset inequalities. We show that Benders inequalities associated to extreme rays are metric inequalities. We also show how to strengthen Benders inequalities associated to non-extreme rays to obtain metric inequalities. We show that cutset inequalities are Benders inequalities, but not necessarily metric inequalities. We give a necessary and sufficient condition for a cutset inequality to be a metric inequality. Computational experiments show the effectiveness of strengthening Benders and cutset inequalities to obtain metric inequalities.

**Keywords** Multicommodity capacitated network design · Benders decomposition · Metric inequalities · Cutset inequalities

A.M. Costa
Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo, CP 668, São Carlos 13560-970, São Paulo, Brazil

J.-F. Cordeau (✉)
Chaire de recherche du Canada en distributique and Centre de recherche sur les transports, HEC Montréal, 3000 chemin de la Côte-Sainte-Catherine, Montréal, Canada H3T 2A7
e-mail: jean-francois.cordeau@hec.ca

B. Gendron
Département d'informatique et de recherche opérationnelle, and Centre de recherche sur les transports, Université de Montréal, C.P. 6128, succ. Centre-ville, Montréal, Canada H3C 3J7

## 1 Introduction

Network design problems have received significant attention in the literature for more than two decades, as evidenced by numerous reviews [3, 12, 18, 19]. In particular, multicommodity capacitated network design problems have been widely studied [1, 2, 9, 15, 21].

A typical network design problem is defined over a directed network $G = (N, A, K)$ with node set $N$, arc set $A$, and commodity set $K$, each commodity $k$ being represented by a triplet $(O(k), D(k), d_k)$, where $O(k)$ is the origin node, $D(k)$ is the destination node and $d_k > 0$ is the demand to be routed between $O(k)$ and $D(k)$. For each commodity $k$, we assume that there exists at least one path from $O(k)$ to $D(k)$ and we define the connected subnetwork $G^k = (N^k, A^k)$ that contains only nodes and arcs that belong to some path from $O(k)$ to $D(k)$. Given a nonconvex objective function, the problem is to find a minimum cost solution that satisfies the demands for all commodities, often subject to additional constraints, such as budgetary limits and topological restrictions.

In this context, many solution algorithms make use of the *multicommodity network flow subproblem*: given a capacity vector $\omega = (\omega_{ij})_{(i,j)\in A} \geq 0$, we wish to identify a multicommodity flow vector $x = (x_{ij}^k)_{(i,j)\in A^k}^{k\in K} \geq 0$ satisfying

$$\sum_{j\in N_i^k(+)} x_{ij}^k - \sum_{j\in N_i^k(-)} x_{ji}^k = \begin{cases} d_k, & i = O(k), \\ 0, & i \neq O(k), D(k), \\ -d_k, & i = D(k), \end{cases} \quad \forall i \in N^k,\ \forall k \in K, \tag{1}$$

$$\sum_{k\in K} x_{ij}^k \leq \omega_{ij}, \quad \forall (i,j) \in A, \tag{2}$$

where $N_i^k(+) = \{j \in N^k | (i,j) \in A^k\}$ and $N_i^k(-) = \{j \in N^k | (j,i) \in A^k\}$. In this formulation, the value $\omega_{ij}$ represents whether or not a connection is established on arc $(i, j)$, and what is the capacity available to carry the demands through this arc. We say that $\omega$ is a *feasible* capacity vector if and only if there exists a multicommodity flow $x \geq 0$ satisfying (1) and (2).

For each commodity $k$, one of the flow conservation equations (1) is redundant. Hence, if we introduce dual variables $\pi = (\pi_i^k)_{i\in N^k}^{k\in K}$ associated to these equations, we can assume $\pi_{O(k)}^k = 0$ for each $k \in K$. Introducing also dual variables $\alpha = (\alpha_{ij})_{(i,j)\in A} \geq 0$ associated to the capacity constraints (2), we have, by strong duality in linear programming, that $\omega$ is feasible if and only if the following dual problem, $D(\omega)$, is bounded:

$$Z(\omega) = \text{Maximize} \sum_{k\in K} d_k \pi_{D(k)}^k - \sum_{(i,j)\in A} \omega_{ij}\alpha_{ij} \tag{3}$$

$$\text{subject to} \quad \pi_j^k - \pi_i^k - \alpha_{ij} \leq 0, \quad \forall (i,j) \in A^k,\ \forall k \in K. \tag{4}$$

Note that the feasible domain for $D(\omega)$, $P_D$, is a full-dimensional polyhedral cone. Any solution in $P_D$ is called a *ray* and we know, by Minkowski's theorem, that every such ray can be expressed as a nonnegative linear combination of the extreme rays of $P_D$ (a ray is *extreme* if we cannot express it as the sum of other rays). We denote by $E(P_D)$ the set of extreme rays of $P_D$.

In this paper, we study the relationships between four well-known classes of inequalities characterizing the feasibility of a capacity vector $\omega$:

- *Benders inequalities.* The dual problem $D(\omega)$ is bounded if and only if $Z(\omega) \leq 0$ if and only if

$$\sum_{k\in K} d_k \pi^k_{D(k)} - \sum_{(i,j)\in A} \omega_{ij}\alpha_{ij} \leq 0, \quad \forall(\pi,\alpha) \in P_D. \tag{5}$$

  We call these the Benders inequalities, since they are used as cuts in the famous Benders decomposition algorithm [5]. In case $\alpha = 0$, we obtain a so-called *trivial* Benders inequality.
- *Benders inequalities associated to extreme rays.* Since any ray can be expressed as a nonnegative linear combination of the extreme rays of $P_D$, we can restrict Benders inequalities to the set of extreme rays and yet obtain a characterization of the feasibility of $\omega$:

$$\sum_{k\in K} d_k \pi^k_{D(k)} - \sum_{(i,j)\in A} \omega_{ij}\alpha_{ij} \leq 0, \quad \forall(\pi,\alpha) \in E(P_D). \tag{6}$$

- *Metric inequalities.* The dual problem, $D(\omega)$, is bounded if and only if, for any $\alpha \geq 0$, we have

$$\sum_{(i,j)\in A} \omega_{ij}\alpha_{ij} \geq \max\left\{\sum_{k\in K} d_k \pi^k_{D(k)} | \pi^k_j - \pi^k_i \leq \alpha_{ij},\ \forall(i,j) \in A^k,\ \forall k \in K\right\}. \tag{7}$$

  The problem on the right-hand side of this inequality is the dual of a shortest path problem, with arc distances equal to $\alpha \geq 0$. Assume we solve this problem and obtain the shortest path lengths $\pi(\alpha) = (\pi^k_i(\alpha))^{k\in K}_{i\in N^k}$ between $O(k)$ and $i$, for each node $i \in N^k$ and commodity $k \in K$. We then have the following metric inequalities:

$$\sum_{k\in K} d_k \pi^k_{D(k)}(\alpha) - \sum_{(i,j)\in A} \omega_{ij}\alpha_{ij} \leq 0, \quad \forall\alpha \geq 0. \tag{8}$$

  Note that $(\pi(\alpha), \alpha) \in P_D$; hence, metric inequalities are a special case of Benders inequalities.
- *Cutset inequalities.* Let $S \subset N$ be a non-empty subset of $N$ and $\overline{S} = N \setminus S$ its complement; $(S, \overline{S})$ is a cutset, i.e., the set of arcs that connect a node in $S$ to a node in $\overline{S}$. Also, let $K(S, \overline{S}) \subseteq K$ be the set of commodities having their origin in $S$ and their destination in $\overline{S}$ and $d_{(S,\overline{S})} = \sum_{k\in K(S,\overline{S})} d^k$. We then define the following cutset inequalities:

$$\sum_{(i,j)\in(S,\overline{S})} \omega_{ij} \geq d_{(S,\overline{S})}, \quad \forall S \subset N,\ S \neq \emptyset. \tag{9}$$

These inequalities simply mean that there should be enough capacity on the arcs of any cutset to satisfy the demands that must be routed through that cutset. They are often used in branch-and-cut algorithms for multicommodity capacitated network design problems [4, 6].

Clearly, cutset inequalities are necessary for a capacity vector $\omega$ to be feasible, but it is well-known that they are not sufficient in general (they are in the single-commodity case, due to the max flow-min cut theorem). As we have just seen, Benders inequalities, Benders inequalities associated to extreme rays, and metric inequalities are both necessary and sufficient to characterize a feasible capacity vector $\omega$. In this sense, the three classes of inequalities are equivalent. However, when it comes to use these inequalities in a solution algorithm, the question of which class to use is not necessarily a trivial one, because even if one class of inequalities is a subclass of another, it might be easier to generate inequalities in the larger class. For example, in a Benders decomposition algorithm, generating only Benders cuts associated to extreme rays is not always better than using other Benders inequalities, as suggested by our computational results, presented in Sect. 4.

To compare inequalities, we use the standard *dominance* criterion. Given a polyhedron $P = \{u|Au \leq b\}$ and two valid inequalities $\lambda^1 u \leq \lambda_0^1$ and $\lambda^2 u \leq \lambda_0^2$ for $P$, we say that the first dominates the second if $\lambda^2 u \leq \lambda^1 u$, $\forall u \in P$ and $\lambda_0^1 \leq \lambda_0^2$. Furthermore, if there exists $u \in P$ such that $\lambda^2 u < \lambda^1 u$, the dominance is strict.

In this paper, we show the following properties:

1. Every Benders inequality associated to an extreme ray is a metric inequality.
2. For a given $\alpha \geq 0$, the corresponding metric inequality dominates all Benders inequalities associated to that particular $\alpha$; as a consequence, any Benders inequality can be *strenghtened* to a metric inequality by solving the shortest path problem in (7).
3. Cutset inequalities are a subclass of Benders inequalities, but are not necessarily metric inequalities (contrary to a common belief).
4. We give a necessary and sufficient condition for a cutset inequality to be a metric inequality.

Our computational experiments, performed on instances of the multicommodity capacitated fixed-charge network design problem, illustrate the following results:

1. On many instances, we show that a large proportion of cutset inequalities can be strenghtened to metric inequalities.
2. For a standard Benders decomposition algorithm, we show that restricting the method to Benders inequalities associated to extreme rays is not necessarily more efficient than generating cuts associated to any ray, extreme or not (contrary to a common belief).
3. For the same algorithm, we show, however, that strengthening Benders cuts to generate only metric inequalities is computationally more efficient than using Benders inequalities generated by a linear programming solver.

This paper is organized as follows. In Sect. 2, we present our results concerning metric inequalities, comparing them to Benders inequalities. Section 3 is dedicated to our developments on cutset inequalities. Section 4 presents our computational results, while Sect. 5 ends this paper with some conclusions.

## 2 Benders and metric inequalities

In this section, we show two results that allow us to compare Benders and metric inequalities, and we illustrate these results on a simple example.

We already know that Benders inequalities associated to extreme rays is a subclass of Benders inequalities. Our first result states that Benders inequalities associated to extreme rays is also a subclass of metric inequalities. To prove this result, we use the following simple lemma.

**Lemma 1** *If* $(\pi, \alpha)$ *is a ray of* $P_D$*, then* $\pi_h^k \leq \pi_h^k(\alpha), \forall h \in N^k, \forall k \in K$.

*Proof* Let $k \in K$ and $P_h^k$ a shortest path (with respect to $\alpha$) from $O(k)$ to $h \in N^k$. We then have $\pi_h^k = \sum_{(i,j)\in P_h^k}(\pi_j^k - \pi_i^k) \leq \sum_{(i,j)\in P_h^k} \alpha_{ij} = \pi_h^k(\alpha)$. □

**Proposition 1** *Any non-trivial Benders inequality associated to an extreme ray is a metric inequality.*

*Proof* Let $(\pi, \alpha)$ an extreme ray of $P_D$ associated to a non-trivial Benders inequality. Assume that this inequality is not a metric inequality. This implies that there exists some commodity $l \in K$ such that $\pi_{D(l)}^l \neq \pi_{D(l)}^l(\alpha)$. By Lemma 1, we must have $\pi_{D(l)}^l < \pi_{D(l)}^l(\alpha)$. This implies that, on every path $P^l$ from $O(l)$ to $D(l)$, there exists an arc $(i, j)$ such that $\pi_j^l - \pi_i^l - \alpha_{ij} < 0$, called a *non-binding arc*; otherwise, we would have a path $Q^l$ such that $\pi_j^l - \pi_i^l - \alpha_{ij} = 0$ on every arc $(i, j) \in Q^l$, and $\pi_{D(l)}^l = \sum_{(i,j)\in Q^l}(\pi_j^l - \pi_i^l) = \sum_{(i,j)\in Q^l} \alpha_{ij} \geq \pi_{D(l)}^l(\alpha)$, a contradiction.

Using this property, we now construct a set $T^l \subset N^l$ as follows: we visit each path $P^l$ from $O(l)$ to $D(l)$, starting from $O(l)$, until we encounter the first non-binding arc $(i, j)$; we then add to $T^l$ (initialized to $\emptyset$) all nodes on $P^l$ from $j$ to $D(l)$. Clearly, $O(l) \notin T^l$ and $D(l) \in T^l$, i.e., $T^l$ defines a cutset. Moreover, it is not possible to have an arc $(i, j)$ such that $i \in T^l$ and $j \notin T^l$. Indeed, $i \in T^l$ implies that there is a path from $O(l)$ to $i$ containing a non-binding arc and $j \in N^l$ implies that there is a path from $j$ to $D(l)$. Connecting these two paths through arc $(i, j)$, we would obtain a path from $O(l)$ to $D(l)$ containing a non-binding arc preceding node $j$. Hence, if $i \in T^l$, we must have $j \in T^l$.

Let $\delta = \min_{i\notin T^l, j\in T^l}\{-\pi_j^l + \pi_i^l + \alpha_{ij}\} > 0$, and define $(\overline{\pi}, \alpha)$ and $(\widetilde{\pi}, 0)$ as follows:

$\overline{\pi} = \pi$, except for $\overline{\pi}_i^l = \pi_i^l + \delta, \ \forall i \in T^l$;
$\widetilde{\pi} = 0$, except for $\widetilde{\pi}_i^l = -\delta, \ \forall i \in T^l$.

We then have:

(1) $(\overline{\pi}, \alpha)$ is a ray, since for any $(i, j) \in A^l$, there are three possible cases:
   (a) $i \in T^l$ and $j \in T^l$: $\overline{\pi}_j^l - \overline{\pi}_i^l - \alpha_{ij} = (\pi_j^l + \delta) - (\pi_i^l + \delta) - \alpha_{ij} = \pi_j^l - \pi_i^l - \alpha_{ij} \leq 0$;
   (b) $i \notin T^l$ and $j \notin T^l$: $\overline{\pi}_j^l - \overline{\pi}_i^l - \alpha_{ij} = \pi_j^l - \pi_i^l - \alpha_{ij} \leq 0$;

(c) $i \notin T^l$ and $j \in T^l$: $\overline{\pi}^l_j - \overline{\pi}^l_i - \alpha_{ij} = (\pi^l_j + \delta) - \pi^l_i - \alpha_{ij} \leq \pi^l_j + (-\pi^l_j + \pi^l_i + \alpha_{ij}) - \pi^l_i - \alpha_{ij} = 0$.

(2) $(\widetilde{\pi}, 0)$ is a ray, since for any $(i, j) \in A^l$, there are three possible cases:
  (a) $i \in T^l$ and $j \in T^l$: $\widetilde{\pi}^l_j - \widetilde{\pi}^l_i = -\delta - (-\delta) = 0$;
  (b) $i \notin T^l$ and $j \notin T^l$: $\widetilde{\pi}^l_j - \widetilde{\pi}^l_i = 0 - 0 = 0$;
  (c) $i \notin T^l$ and $j \in T^l$: $\widetilde{\pi}^l_j - \widetilde{\pi}^l_i = -\delta - 0 < 0$.

(3) $(\pi, \alpha) = (\overline{\pi}, \alpha) + (\widetilde{\pi}, 0)$, which contradicts the hypothesis that $(\pi, \alpha)$ is an extreme ray. □

Even though Benders inequalities associated to extreme rays subsume the two other classes of inequalities, some solution algorithms might generate Benders inequalities that are not necessarily associated to extreme rays. If we let $(\pi, \alpha)$ a ray (not necessarily extreme) associated to a Benders inequality, one might solve a shortest path problem with respect to $\alpha$ and derive a metric inequality associated to the ray $(\pi(\alpha), \alpha)$. Our next result states that we then obtain a metric inequality that always dominates the original Benders inequality.

**Proposition 2** *Any Benders inequality associated to ray* $(\pi, \alpha)$ *is dominated by the metric inequality associated to ray* $(\pi(\alpha), \alpha)$. *The dominance is strict if and only if there exists* $l \in K$ *such that* $\pi^l_{D(l)} < \pi^l_{D(l)}(\alpha)$.

*Proof* Using the definition of dominance given in the introduction, it suffices to show that

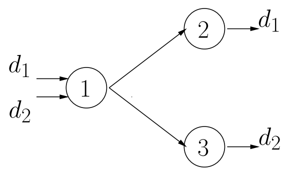
$$\sum_{k \in K} d_k \pi^k_{D(k)} - \sum_{(i,j) \in A} \omega_{ij} \alpha_{ij} \leq \sum_{k \in K} d_k \pi^k_{D(k)}(\alpha) - \sum_{(i,j) \in A} \omega_{ij} \alpha_{ij}.$$

But, this is immediate since, for each $k \in K$, $d^k > 0$ and $\pi^k_{D(k)} \leq \pi^k_{D(k)}(\alpha)$, by Lemma 1. It also follows immediately that the dominance is strict if and only if there exists $l \in K$ such that $\pi^l_{D(l)} < \pi^l_{D(l)}(\alpha)$. □

*Example 1* Figure 1 shows a network with three vertices, two arcs and two commodities, with demands $d_1$ and $d_2$, having their origins at node 1 and their destinations at nodes 2 and 3, respectively. Once we fix to 0 the dual variables associated to redundant constraints of the multicommodity flow subproblem (i.e., flow conservation equations for commodity 1, nodes 1 and 3, and for commodity 2, nodes 1 and 2) we obtain the following full-dimensional dual polyhedron:

$$P_D = \{(\pi, \alpha) = ((\pi^1_2, \pi^2_3), (\alpha_{12}, \alpha_{13})) | \pi^1_2 - \alpha_{12} \leq 0,\ \pi^2_3 - \alpha_{13} \leq 0,\ \alpha_{12}, \alpha_{13} \geq 0\}.$$

**Fig. 1** Network of Example 1



We can easily enumerate the extreme rays of $P_D$:

$$(\pi(1),\alpha(1)) = ((1,0),(1,0));$$
$$(\pi(2),\alpha(2)) = ((0,1),(0,1));$$
$$(\pi(3),\alpha(3)) = ((-1,0),(0,0));$$
$$(\pi(4),\alpha(4)) = ((0,-1),(0,0)).$$

There are two non-trivial Benders inequalities associated to these extreme rays:

$$\omega_{12} \geq d_1 \quad (\text{associated to } (\pi(1),\alpha(1)));$$
$$\omega_{13} \geq d_2 \quad (\text{associated to } (\pi(2),\alpha(2))).$$

If we take $\alpha = (1,1)$, we obtain the metric inequality $\omega_{12} + \omega_{13} \geq d_1 + d_2$, associated to the ray $((1,1),(1,1)) = (\pi(1),\alpha(1)) + (\pi(2),\alpha(2))$, i.e., this metric inequality is obtained by aggregation of the two non-trivial Benders inequalities associated to extreme rays. In other words, the combination of the two Benders inequalities associated to extreme rays dominates this metric inequality, but none of them, individually, dominates it. Note that there are other Benders inequalities associated to $\alpha = (1,1)$, for instance $\omega_{12} + \omega_{13} \geq d_1$ (associated to the ray $((1,0),(1,1))$) and $\omega_{12} + \omega_{13} \geq d_2$ (associated to the ray $((0,1),(1,1))$), which are not metric inequalities and are dominated by $\omega_{12} + \omega_{13} \geq d_1 + d_2$.

Suppose we perform two iterative solution algorithms. The first one generates, at each iteration, one Benders inequality associated to an extreme ray. The second algorithm generates, at each iteration, one Benders inequality and strengthens it to a metric inequality by solving a shortest path problem with $\alpha$ fixed at its value in the generated Benders inequality. Assume that, at some iteration, both algorithms solve the multicommodity flow subproblem with $\overline{\omega}$ satisfying $\overline{\omega}_{12} < d_1$ and $\overline{\omega}_{13} < d_2$. Also assume that $d_2 - \overline{\omega}_{13} > d_1 - \overline{\omega}_{12}$. Suppose the first algorithm generates the Benders inequality $\omega_{13} \geq d_2$. The second algorithm first generates the Benders inequality $\omega_{12} + \omega_{13} \geq d_2$ and then strengthens it to the metric inequality $\omega_{12} + \omega_{13} \geq d_1 + d_2$. This inequality is violated by $\overline{\omega}$ by the quantity $(d_2 - \overline{\omega}_{13}) + (d_1 - \overline{\omega}_{12}) > d_2 - \overline{\omega}_{13}$, the amount by which the Benders inequality $\omega_{13} \geq d_2$ is violated.

This example shows that, even if we select the best Benders inequality associated to an extreme ray, we might obtain a better metric inequality associated to a non-extreme ray. Thus, even though we can restrict ourselves to Benders inequalities associated to extreme rays, it might be interesting from a computational point of view to generate other Benders inequalities, in particular metric inequalities. Our computational results, presented in Sect. 4, confirm this observation.

A final remark is that one could certainly try to generate metric inequalities directly instead of generating Benders inequalities, then strengthening them to obtain metric inequalities. Indeed, [9] have proposed a cutting plane algorithm based on metric inequalities for a multicommodity capacitated network design problem. In their algorithm, a heuristic method is used to generate violated metric inequalities. In general, however, it is harder to derive violated metric inequalities than to obtain violated Benders inequalities, in which case a state-of-the-art linear programming software package can be used to solve the multicommodity flow subproblem.

## 3 Cutset inequalities

In this section, we investigate the relationships between cutset inequalities and the three other classes of inequalities defined in the introduction. We first show that cutset inequalities are a subclass of Benders inequalities. To this purpose, we need to introduce, for any $S \subset N$, $S \neq \emptyset$, the ray $(\pi(S), \alpha(S))$ defined as follows:

$$\pi_i^k(S) = \begin{cases} 1, & \text{if } i \in \overline{S},\ O(k) \in S \text{ and } D(k) \in \overline{S},\ \forall i \in N^k,\ \forall k \in K, \\ 0, & \text{otherwise,} \end{cases}$$

$$\alpha_{ij}(S) = \begin{cases} 1, & \text{if } i \in S \text{ and } j \in \overline{S},\ \forall (i,j) \in A, \\ 0, & \text{otherwise.} \end{cases}$$

**Proposition 3** *For any $S \subset N$, $S \neq \emptyset$, the cutset inequality associated to $S$ corresponds to the Benders inequality associated to the ray $(\pi(S), \alpha(S))$.*

*Proof* By definition of $(\pi(S), \alpha(S))$, we have $\pi^k_{D(k)}(S) = 1$ for each commodity $k \in K(S, \overline{S})$ and $\alpha_{ij}(S) = 1$ if and only if $(i, j) \in (S, \overline{S})$ for each arc $(i, j)$, which implies:

$$\begin{aligned} \sum_{k\in K} d_k \pi^k_{D(k)}(S) - \sum_{(i,j)\in A} \omega_{ij}\alpha_{ij}(S) &= \sum_{k\in K(S,\overline{S})} d_k - \sum_{(i,j)\in(S,\overline{S})} \omega_{ij} \\ &= d_{(S,\overline{S})} - \sum_{(i,j)\in(S,\overline{S})} \omega_{ij}. \end{aligned}$$

The Benders inequality $\sum_{k\in K} d_k \pi^k_{D(k)}(S) - \sum_{(i,j)\in A} \omega_{ij}\alpha_{ij}(S) \leq 0$ is thus equivalent to the cutset inequality $d_{(S,\overline{S})} - \sum_{(i,j)\in(S,\overline{S})} \omega_{ij} \leq 0$. □

In general, a cutset inequality is not necessarily a metric inequality (consequently, it is not necessarily a Benders inequality associated to an extreme ray). The next proposition gives a necessary and sufficient condition for a cutset inequality to be a metric inequality.

**Proposition 4** *For any $S \subset N$, $S \neq \emptyset$, the cutset inequality associated to $S$ is a metric inequality if and only if for each $k \in K$*

(a) *if $O(k) \in S$ and $D(k) \in \overline{S}$, there exists a path from $O(k)$ to $D(k)$ that crosses $(S, \overline{S})$ only once*;
(b) *if $O(k) \in \overline{S}$ or $D(k) \in S$, there exists a path from $O(k)$ to $D(k)$ that never crosses $(S, \overline{S})$.*

*Proof* Assume we solve the shortest path problem associated to the distances $\alpha(S)$ and obtain the shortest path lengths $\pi(\alpha(S))$. By definition of $\alpha(S)$, for each $k \in K$, there are three possible outcomes:

(1) $\pi^k_{D(k)}(\alpha(S)) = 0$, which means that there exists a path from $O(k)$ to $D(k)$ that never crosses $(S, \overline{S})$;
(2) $\pi^k_{D(k)}(\alpha(S)) = 1$, which means that there exists a path from $O(k)$ to $D(k)$ that crosses $(S, \overline{S})$ only once;
(3) $\pi^k_{D(k)}(\alpha(S)) > 1$, which means that there is no path from $O(k)$ to $D(k)$ that never crosses $(S, \overline{S})$ or that crosses $(S, \overline{S})$ only once.

Now, let us assume that the cutset inequality associated to $S \subset N$, $S \neq \emptyset$, is a metric inequality. For any $k \in K$, we then have $\pi^k_{D(k)}(\alpha(S)) = \pi^k_{D(k)}(S)$, and

(a) if $O(k) \in S$ and $D(k) \in \overline{S}$, then, by definition of $\pi(S)$, we have $1 = \pi^k_{D(k)}(S) = \pi^k_{D(k)}(\alpha(S))$, which implies that there exists a path from $O(k)$ to $D(k)$ that crosses $(S, \overline{S})$ only once;
(b) if $O(k) \in \overline{S}$ or $D(k) \in S$, then, by definition of $\pi(S)$, we have $0 = \pi^k_{D(k)}(S) = \pi^k_{D(k)}(\alpha(S))$, which implies that there exists a path from $O(k)$ to $D(k)$ that never crosses $(S, \overline{S})$.

Conversely, let us assume that, for each $k \in K$, we have either

(a) $O(k) \in S$ and $D(k) \in \overline{S}$, in which case there exists a path from $O(k)$ to $D(k)$ that crosses $(S, \overline{S})$ only once, which implies that $\pi^k_{D(k)}(\alpha(S)) = 1 = \pi^k_{D(k)}(S)$, by definition of $\pi(S)$; or
(b) $O(k) \in \overline{S}$ or $D(k) \in S$, in which case there exists a path from $O(k)$ to $D(k)$ that never crosses $(S, \overline{S})$, which implies $\pi^k_{D(k)}(\alpha(S)) = 0 = \pi^k_{D(k)}(S)$, by definition of $\pi(S)$.

Thus, for each $k \in K$, $\pi^k_{D(k)}(\alpha(S)) = \pi^k_{D(k)}(S)$, and the cutset inequality associated to $S$ is a metric inequality. □

*Example 2* Consider the network depicted in Fig. 2, with four nodes (1, 2, 3 and 4) and 3 arcs ($(1, 2)$, $(3, 4)$ and $(4, 1)$). There are two commodities, 1 and 2, with demands $d_1$ and $d_2$, respectively, and $O(1) = 4$, $O(2) = 3$, $D(1) = D(2) = 2$.

Consider the cutset $S = \{1, 3\}$. The subnetworks $G^k = (N^k, A^k)$ for $k = 1, 2$ are shown in Fig. 3, along with the ray $(\pi(S), \alpha(S))$ associated to $S$. For each commod-

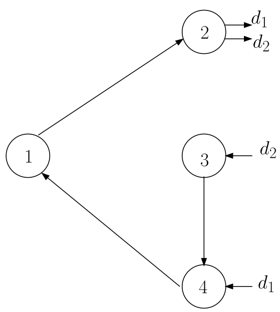



**Fig. 2** Network for Example 2—4 nodes, 3 arcs and 2 commodities

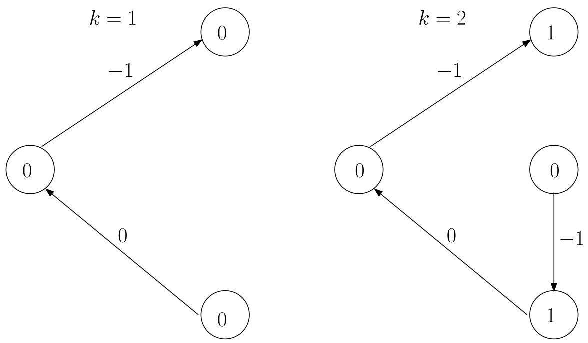



**Fig. 3** Subnetworks for each commodity and ray associated to the cutset inequality for Example 2

ity $k$, the value inside each node $i$ corresponds to $\pi_i^k(S)$, while the value close to each arc $(i, j)$ is $\alpha_{ij}(S)$. This ray is associated to the Benders inequality $\omega_{12} + \omega_{34} \geq d_2$.

The length of the shortest path (with respect to $\alpha(S)$) between $O(2)$ and $D(2)$ is equal to 2. Moreover, even though neither $O(1)$ nor $D(1)$ are in set $S$, the shortest path length between $O(1)$ and $D(1)$ is equal to 1. Therefore, strenghtening the cutset inequality yields the metric inequality $\omega_{12} + \omega_{34} \geq d_1 + 2d_2$, which dominates the original cutset inequality.

The intuition behind this strenghtening is very simple. Observe that in Example 2 the only path between $O(2)$ $(\in S)$ and $D(2)$ $(\in \overline{S})$ must use more than one arc from the cutset (in this case, arcs $(3, 4)$ and $(1, 2)$), because it needs to reenter set $S$. Therefore, $\pi^2_{D(2)}(\alpha(S))$ actually represents the number of times *an arc crossing the cutset* has to be used in order for the commodity to leave its origin $O(2)$ and reach its destination $D(2)$. The cutset inequality does not take this information into account (as if a single use of the cutset arcs would always be enough) and therefore presents a worse estimation of the needed capacity. This strengthening also considers commodities for which neither the origin nor the destination is in set $S$, but still need to use some capacity across the cutset, as is the case for commodity 1.

A final remark on Example 2: it is interesting to note that the same metric inequality can be obtained as the sum of two cutset inequalities: one associated with $S = \{3\}$, which yields the inequality $\omega_{34} \geq d_2$, and the other associated to $S = \{1, 3, 4\}$, which yields the inequality $\omega_{12} \geq d_1 + d_2$. There are cases, however, in which one cannot obtain the strengthened metric inequality by adding up cutset inequalities, as shown in the next example.

*Example 3* Consider the network of Example 2 with the addition of arc $(3, 2)$. Again, we consider the cutset defined by $S = \{1, 3\}$. The subnetworks for the two commodities and the ray associated to this cutset inequality are shown in Fig. 4. The cutset inequality is $\omega_{12} + \omega_{34} + \omega_{32} \geq d_2$. The strengthened metric inequality is $\omega_{12} + \omega_{34} + \omega_{32} \geq d_1 + d_2$, which cannot be obtained as the sum of other cutset inequalities. Note, however, that it is dominated by the cutset inequality associated to $S = \{1, 3, 4\}$, which is $\omega_{12} + \omega_{32} \geq d_1 + d_2$.

The last two examples show that strenghtened metric inequalities can be dominated by other cutset inequalities. In spite of this fact, strenghtening cutset inequalities is interesting, because solution algorithms usually make use of heuristic methods

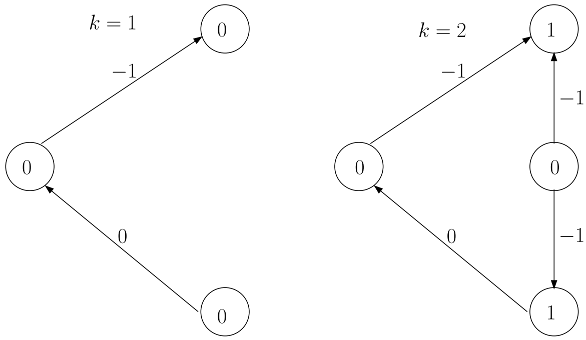



**Fig. 4** Subnetworks for each commodity and ray associated to the cutset inequality for Example 3

to generate cutset inequalities. Hence, they do not guarantee that the best cutsets are generated. The proposed strenghtening can be useful in that context, as confirmed by our computational experiments, reported in Sect. 4.1.

## 4 Computational results

To perform our computational experiments, we use instances of the *multicommodity capacitated fixed-charge network design problem* (MCFND), for which several exact and approximate solution algorithms have been proposed [7, 8, 10, 11, 13, 14, 16, 17, 20]. To each arc $(i, j) \in A$ is associated a binary variable $y_{ij}$ taking value 1 if and only if the arc is chosen, and continuous variables $x^k_{ij}$, $k \in K$, representing the flow of commodity $k$ on the arc. Parameters $f_{ij}$ and $c^k_{ij}$ are, respectively, the fixed cost for opening arc $(i, j)$ and the variable cost associated with the flow of one unit of commodity $k$ through arc $(i, j)$, while $u_{ij}$ is the capacity of arc $(i, j)$ when it is open. With this notation, one can write the MCFND as follows:

$$\text{Minimize} \sum_{(i,j)\in A} \left( f_{ij} y_{ij} + \sum_{k\in K} c^k_{ij} x^k_{ij} \right) \tag{10}$$

subject to

$$\sum_{j\in N_i^+} x^k_{ij} - \sum_{j\in N_i^-} x^k_{ji} = \begin{cases} d_k, & i = O(k), \\ 0, & i \notin \{O(k), D(k)\}, \\ -d_k, & i = D(k), \end{cases} \quad \forall i \in N,\ \forall k \in K, \tag{11}$$

$$\sum_{k\in K} x^k_{ij} \le u_{ij} y_{ij}, \quad \forall (i, j) \in A, \tag{12}$$

$$x^k_{ij} \ge 0, \qquad \forall (i, j) \in A,\ \forall k \in K, \tag{13}$$

$$y_{ij} \in \{0, 1\}, \qquad \forall (i, j) \in A. \tag{14}$$

Our computational experiments are based on three sets of instances widely used in the literature on the MCFND. These instances contain from 10 to 100 nodes, 35 to 687 arcs and 10 to 400 commodities. They are described in detail by Ghamlouche, Crainic and Gendreau [14].

In the next subsection, we present results obtained on cutset inequalities generated by an enumeration of all cutsets, up to a limited cardinality. We are interested in the proportion of cutset inequalities that are strenghtened, as well as by the improvement obtained by the strenghtened metric inequalities. Subsequently, in Sect. 4.2, we analyze the results obtained by a Benders algorithm applied to the MCFND. We investigate, in particular, the relative performance of three variants of the algorithm: one that generates Benders inequalities associated to rays that are not necessarily extreme; a second variant that generates only Benders inequalities associated to extreme rays; and a third variant that generates Benders inequalities and strenghtens them to obtain only metric inequalities.

## 4.1 Cutset inequalities

In the special case of cutset inequalities, we have considered for each instance all sets $S$, with $|S| \leq \Psi$, $\Psi$ being an integer which depends on the number of nodes in the graph (see Tables 1 and 2). All the associated cutsets have been generated and tested. In Table 1 the results are shown for the first group of instances (group R), while Table 2 shows the results obtained for the third group (group C+). For the second group (group C), considering sets with $|S| \leq 4$, no cutset inequality has been strengthened. In the tables, for each instance, we show the following

**Table 1** Effect of strengthening on cutset inequalities (group R)

| Inst. | $(\|N\|, \|A\|, \|K\|)$ | $\Psi$ | Nb. cutsets | Nb. strength | Max strength (%) | Avg strength (%) |
|---|---|---|---|---|---|---|
| R01.1 | (10,35,10) | 5 | 637 | 32 | 42.11 | 32.35 |
| R02.1 | (10,35,25) | 5 | 637 | 76 | 34.97 | 19.41 |
| R03.1 | (10,35,50) | 5 | 637 | 63 | 24.02 | 10.51 |
| R04.1 | (10,60,10) | 5 | 637 | 0 | – | – |
| R05.1 | (10,60,25) | 5 | 637 | 0 | – | – |
| R06.1 | (10,60,50) | 5 | 637 | 0 | – | – |
| R07.1 | (10,82,80) | 5 | 637 | 0 | – | – |
| R08.1 | (10,83,25) | 5 | 637 | 0 | – | – |
| R09.1 | (10,83,50) | 5 | 637 | 0 | – | – |
| R10.1 | (20,120,40) | 5 | 6195 | 2 | 14.70 | 14.18 |
| R11.1 | (20,120,100) | 5 | 6195 | 48 | 12.26 | 4.28 |
| R12.1 | (20,120,200) | 5 | 6195 | 67 | 4.98 | 2.92 |
| R13.1 | (20,220,40) | 5 | 6195 | 0 | – | – |
| R14.1 | (20,220,100) | 5 | 6195 | 0 | – | – |
| R15.1 | (20,220,200) | 5 | 6195 | 0 | – | – |
| R16.1 | (20,314,40) | 5 | 6195 | 0 | – | – |
| R17.1 | (20,318,100) | 5 | 6195 | 0 | – | – |
| R18.1 | (20,315,200) | 5 | 6195 | 0 | – | – |

**Table 2** Effect of strengthening on cutset inequalities (group C+)

| Inst. | $(\|N\|, \|A\|, \|K\|)$ | $\Psi$ | Nb. cutsets | Nb. strength | Max strength (%) | Avg strength (%) |
|---|---|---|---|---|---|---|
| c25_100_10_F_L_5 | (25,100,10) | 4 | 15.275 | 22 | 100.00 | 45.71 |
| c25_100_10_F_T_5 | (25,100,10) | 4 | 15.275 | 41 | 100.00 | 88.09 |
| c25_100_10_V_L_5 | (25,100,10) | 4 | 15.275 | 160 | 100.00 | 65.77 |
| c25_100_30_F_L_5 | (25,100,30) | 4 | 15.275 | 143 | 50.00 | 18.45 |
| c25_100_30_F_T_5 | (25,100,30) | 4 | 15.275 | 868 | 100.00 | 25.07 |
| c25_100_30_V_T_5 | (25,100,30) | 4 | 15.275 | 785 | 100.00 | 29.80 |
| c100_400_10_F_L_10 | (100,400,10) | 3 | 166.750 | 3 | 100.00 | 84.84 |
| c100_400_10_F_T_10 | (100,400,10) | 3 | 166.750 | 6 | 100.00 | 60.73 |
| c100_400_10_V_L_10 | (100,400,10) | 3 | 166.750 | 98 | 100.00 | 56.00 |
| c100_400_30_F_L_10 | (100,400,30) | 3 | 166.750 | 595 | 100.00 | 63.47 |
| c100_400_30_F_T_10 | (100,400,30) | 3 | 166.750 | 304 | 100.00 | 75.91 |
| c100_400_30_V_T_10 | (100,400,30) | 3 | 166.750 | 592 | 100.00 | 63.18 |

information:

- $(|N|, |A|, |K|)$—number of nodes, arcs and commodities in the instance;
- $\Psi$—maximum cardinality of the tested cutsets;
- Nb. cutsets—number of cutsets tested;
- Nb. strength—number of cutset inequalities that were strengthened;
- Max strength—the maximum strength obtained;
- Avg strength—the average strength considering all the strengthened inequalities;

where the *strength* is defined as $[\sum_{k \in K} d^k \pi^k_{D(k)}(\alpha) - \sum_{k \in K} d_k \pi^k_{D(k)})]/\sum_{k \in K} d^k \pi^k_{D(k)}(\alpha)$.

The results presented in Tables 1 and 2 indicate that the strengthening is only effective for sparse graphs. This result was expected, since the higher the arc density, the smaller the probability that the conditions given in Proposition 4 are satisfied. Indeed, consider a subset $S$ of the nodes of a complete graph, and its complement $\overline{S}$. If some commodity has its origin in $S$ and its destination in $\overline{S}$, it is always possible to connect the origin to the destination by using a single arc across the cutset. Similarly, if we consider a commodity with its origin in $\overline{S}$ or its destination in $S$, we can reach the destination from the origin by an arc that does not cross the cutset. Thus, in the case of a complete graph, strengthening cutset inequalities is useless, since they are already metric inequalities.

### 4.2 Benders inequalities

We have developed a standard Benders decomposition algorithm which obtains values for the dual variables $(\alpha, \pi)$ by using a state-of-the-art linear programming software package (CPLEX) to solve the multicommodity flow subproblem at each iteration of the algorithm.

The Benders algorithm first relaxes all feasibility and optimality constraints in the master problem. At each iteration, the relaxed master problem provides a lower bound on the optimal solution value of the original problem and a solution $\overline{y}$. These variables define a tentative network, and are used in the dual subproblem.

In case the dual subproblem is unbounded, we obtain a ray (not necessarily extreme) that can be used to generate a violated Benders inequality (also called a feasibility cut). In case the dual subproblem is bounded, the conjunction of the master problem and subproblem solutions is a feasible solution to the original problem (and provides an upper bound). In this case, the extreme point corresponding to a dual optimal solution can be used to generate a so-called optimality cut. The process iterates until the values of the lower and upper bounds coincide.

We have analyzed the number of iterations and the time it takes for this Benders decomposition algorithm to find a first feasible solution. We compare these two figures with and without strengthening the feasibility cuts to obtain metric inequalities. The number of iterations performed before a first feasible solution is found gives a measure of how much we can strengthen the cuts. The comparison of the computation time between the two cases is useful to determine if the time spent in strengthening is rewarded by the gain in the strength of the cuts.

In Tables 3–7, the following results are shown:

- With or Without strength—indicates the use (or not) of strengthening:
  Nb. iter—number of iterations needed to find a first feasible solution;
  Time (s)—time in seconds needed to find a first feasible solution;
- $\neq$ (%)— percentage difference between the equivalent columns with and without strengthening;
- Strength info—information associated with strengthening:
  Nb. strength—number of iterations where the cut was strengthened;
  Max strength—maximum strength obtained (%);
  Avg strength—average strength obtained (%).

Tables 3 and 4 show the results for the first group of instances. We see from these tables that strenghtening the feasibility cuts not only reduces the number of iterations needed to find a feasible solution for most instances, but also the time needed, indicating that the time spent to solve the shortest path problem is rewarded by the strengthening of the cuts. Note that, for each line in the tables, there is actually a set of instances corresponding to the same network, but having different configurations of fixed and variable costs [14]. In the tables, we present the average results.

The results for the second group of instances are presented in Tables 5 and 6. These instances are harder than the first group. Indeed, for about half of these instances (those in Table 6), the Benders algorithm could not find a single feasible solution in the allowed computation time of five hours. Therefore, Table 6 does not compare the difference between the total time to find a feasible solution, but the lower bound upon termination. Although it is a different measure, the lower bounds are also relevant since they somehow indicate the strength of the cuts.

For the instances in Table 5, a mean reduction of 22% in the computation time needed to obtain the first feasible solution was obtained by using strengthening. The results are consistent with those of the first group of instances and indicate that the harder the problem, the more effective the strenghtening seems to be in reducing the total computation time.

An interesting result is obtained in Table 6. In this case, the time limit of 5 hours was not enough to obtain a single feasible solution. However, one observes that the lower bounds obtained by the Benders decomposition algorithm using strengthening were slightly better (mean increase of 4.13%) than those obtained without strenghtening. This indicates that the behavior is similar to the other two groups of instances. Finally, Table 7 shows the results for the last group of instances. Again, these results are similar to those for the other groups.

A final study concerns the relative strength of Benders inequalities and Benders inequalities associated to extreme rays. It is common in the literature to use extreme rays when solving problems by means of a Benders decomposition approach, since they are believed to yield stronger cuts. When solving an unbounded subproblem with CPLEX we had the option of obtaining an extreme ray or a ray (not necessarily extreme). We tested both options on the first group of instances and the results are summarized in Table 8.

These results somehow contradict what is commonly believed, since no clear answer can be given to the question of which is the best option. Our belief is that,

**Table 3** Results for Benders inequalities (group R 10 nodes)

| Instance | Without strength | | With strength | | ≠ (%) | | Strength info | | |
|---|---|---|---|---|---|---|---|---|---|
| | Nb. iter | Time (s) | Nb. iter | Time (s) | Nb. iter | Time | Nb. strength | Max strength | Avg strength |
| R01 | 45.50 | 0.10 | 37.67 | 0.09 | −17.15 | −9.14 | 11.00 | 28.81 | 16.06 |
| R02 | 54.83 | 0.20 | 41.33 | 0.17 | −24.90 | −18.78 | 19.67 | 47.41 | 19.46 |
| R03 | 58.50 | 0.39 | 44.50 | 0.31 | −23.46 | −20.49 | 29.00 | 48.79 | 19.41 |
| R04 | 31.44 | 0.08 | 31.22 | 0.09 | −0.60 | 12.53 | 0.33 | 5.64 | 16.91 |
| R05 | 71.78 | 0.48 | 66.67 | 0.46 | −6.61 | −2.61 | 17.44 | 13.87 | 4.94 |
| R06 | 94.56 | 1.49 | 76.78 | 1.19 | −14.45 | −16.23 | 33.78 | 20.33 | 6.82 |
| R07 | 38.22 | 0.14 | 37.11 | 0.15 | −2.89 | 4.86 | 1.89 | 7.63 | 6.18 |
| R08 | 79.44 | 0.81 | 81.89 | 0.87 | 3.70 | 6.64 | 11.67 | 9.86 | 4.03 |
| R09 | 117.22 | 2.79 | 96.22 | 2.02 | −16.28 | −25.87 | 36.25 | 25.80 | 5.86 |
| Mean | | | | | −10.10 | −6.40 | 17.25 | 20.87 | 9.61 |

**Table 4** Results for Benders inequalities (group R 20 nodes)

| Instance | Without strength | | With strength | | $\neq$ (%) | | Strength info | | |
|---|---|---|---|---|---|---|---|---|---|
| | Nb. iter | Time (s) | Nb. iter | Time (s) | Nb. iter | Time | Nb. strength | Max strength | Avg strength |
| R10 | 265.44 | 10.86 | 225.89 | 9.01 | −12.66 | −13.38 | 112.56 | 32.11 | 8.99 |
| R11 | 282.33 | 48.18 | 204.44 | 34.04 | −24.18 | −27.83 | 136.44 | 62.92 | 14.57 |
| R12 | 209.67 | 57.20 | 141.67 | 36.32 | −30.02 | −30.98 | 116.11 | 57.36 | 16.22 |
| R13 | 431.78 | 97.43 | 399.89 | 91.72 | −8.12 | −6.84 | 117.89 | 17.51 | 3.99 |
| R14 | 760.56 | 483.62 | 597.78 | 393.95 | −17.93 | −17.69 | 339.44 | 37.95 | 5.03 |
| R15 | 542.00 | 708.65 | 353.78 | 483.58 | −27.13 | −23.70 | 255.22 | 62.34 | 7.83 |
| R16 | 389.44 | 159.52 | 351.22 | 144.46 | −9.60 | −4.34 | 80.56 | 17.57 | 3.23 |
| R17 | 1202.00 | 2315.23 | 983.78 | 1958.08 | −18.05 | −15.04 | 519.67 | 35.36 | 4.44 |
| R18 | 978.89 | 3413.53 | 654.00 | 2471.57 | −28.03 | −26.11 | 446.22 | 35.34 | 4.86 |
| Mean | | | | | −19.53 | −18.44 | 236.01 | 39.83 | 7.68 |

**Table 5** Results for Benders inequalities (group C)

| Instance | Without strength | | With strength | | ≠ (%) | | Strength info | | |
|---|---|---|---|---|---|---|---|---|---|
| | Nb. iter | Time (s) | Nb. iter | Time (s) | Nb. iter | Time | Nb. strength | Max strength | Avg strength |
| c33 | 532 | 111.48 | 478 | 103.62 | −10.15 | −7.59 | 140 | 20.13 | 3.88 |
| c35 | 454 | 141.42 | 393 | 120.56 | −13.44 | −17.30 | 88 | 16.54 | 3.40 |
| c36 | 492 | 197.67 | 450 | 168.32 | −8.54 | −17.44 | 85 | 28.55 | 3.90 |
| c37 | 2227 | 8041.07 | 1589 | 6298.42 | −28.65 | −27.67 | 1007 | 44.68 | 3.83 |
| c38 | 2909 | 12738.5 | 1948 | 8906.8 | −33.04 | −43.02 | 1337 | 56.38 | 3.91 |
| c39 | 2469 | 11205.6 | 1655 | 8391.61 | −32.97 | −33.53 | 999 | 47.60 | 4.06 |
| c40 | 2653 | 12295.4 | 1750 | 9123.37 | −34.04 | −34.77 | 1105 | 58.41 | 3.17 |
| c41 | 602 | 325.07 | 557 | 306.45 | −7.48 | −6.08 | 148 | 33.80 | 3.87 |
| c42 | 588 | 331.64 | 553 | 282.48 | −5.95 | −17.40 | 68 | 15.35 | 3.17 |
| c43 | 630 | 588.32 | 643 | 629.94 | 2.06 | 6.61 | 88 | 38.89 | 3.70 |
| c44 | 911 | 893.24 | 793 | 658.8 | −12.95 | −35.59 | 107 | 13.71 | 3.08 |
| c45 | 2959 | 18532.3 | 2179 | 14782 | −26.36 | −25.37 | 1367 | 46.06 | 2.65 |
| c46 | 3024 | 18241.6 | 2206 | 14690.9 | −27.05 | −24.17 | 1300 | 32.49 | 2.60 |
| c47 | 3222 | 21230 | 2193 | 15997.3 | −31.94 | −32.71 | 1294 | 45.43 | 2.61 |
| c48 | 2772 | 15250.2 | 1991 | 12312.1 | −28.17 | −23.86 | 1127 | 52.36 | 2.71 |
| Mean | | | | | −19.91 | −22.66 | 684 | 36.69 | 3.37 |

**Table 6** Results for Benders inequalities (group C—time limit exceeded)

| Instance | Without strength | | With strength | | $\neq$ (%) | Strength info | | |
|---|---|---|---|---|---|---|---|---|
| | Nb. iter | LB | Nb. iter | LB | LB | Nb. strength | Max strength | Avg strength |
| c49 | 849 | 3677.12 | 658 | 3658.39 | −0.51 | 637 | 28.51 | 4.65 |
| c50 | 1077 | 16381.4 | 766 | 16341.7 | −0.24 | 736 | 32.41 | 4.65 |
| c51 | 679 | 2895.79 | 549 | 2930.23 | 1.19 | 515 | 26.01 | 4.13 |
| c52 | 760 | 17175.7 | 555 | 17320.2 | 0.84 | 529 | 26.76 | 4.62 |
| c53 | 1061 | 10920.4 | 560 | 11977 | 9.68 | 553 | 78.01 | 11.56 |
| c54 | 886 | 20131.3 | 472 | 21416.9 | 6.39 | 461 | 49.34 | 9.25 |
| c55 | 769 | 11105.6 | 436 | 11956.4 | 7.66 | 423 | 78.92 | 8.30 |
| c56 | 741 | 22470.3 | 421 | 24242.4 | 7.89 | 410 | 54.51 | 8.90 |
| c57 | 388 | 2595.49 | 344 | 2676.2 | 3.11 | 317 | 24.44 | 4.59 |
| c58 | 432 | 5140.63 | 347 | 5233.47 | 1.81 | 316 | 28.91 | 4.68 |
| c59 | 501 | 2512.28 | 381 | 2538.63 | 1.05 | 352 | 27.53 | 3.80 |
| c60 | 287 | 5271.1 | 236 | 5299.48 | 0.54 | 222 | 21.37 | 3.64 |
| c61 | 690 | 8728.18 | 338 | 9245.4 | 5.93 | 314 | 63.75 | 13.24 |
| c62 | 587 | 17684.2 | 317 | 19029.1 | 7.61 | 306 | 45.13 | 11.41 |
| c63 | 628 | 8467.78 | 325 | 8988.69 | 6.15 | 310 | 49.36 | 9.33 |
| c64 | 492 | 17317.9 | 264 | 18521.3 | 6.95 | 243 | 40.97 | 10.22 |
| Mean | | | | | 4.13 | 415.25 | 42.25 | 7.31 |

although the set of extreme rays is limited (and therefore, common sense would lead us to believe that the associated Benders feasibility cuts are stronger), combining two or more Benders feasibility cuts might sometimes be more effective in cutting the feasible space.

## 5 Conclusion

In this paper, we clarified the relationships between three well-known classes of inequalities used in solution algorithms for multicommodity capacitated network design problems: Benders, metric and cutset inequalities. We have shown that Benders inequalities associated to extreme rays are always metric, but that it can be interesting to generate other Benders inequalities, and strengthen them to obtain metric inequalities. This has been shown in particular on cutset inequalities, a subclass of Benders inequalities, for which we have given a necessary and sufficient condition for them to be metric inequalities. Computational results on a Benders decomposition algorithm have shown that, for some instances, the time needed to find a first feasible solution was reduced by more than 30% when strengthening Benders feasibility cuts to obtain metric inequalities.

Our results are applicable not only in the context of solving the MCFND by a Benders decomposition algorithm. In particular, as mentioned in the Introduction, our theoretical results apply to any network design problem for which feasible solutions can be obtained by solving multicommodity network flow subproblems. Also,

**Table 7** Results for Benders inequalities (group C+)

| Instance | Without strength | | With strength | | $\neq$ (%) | | Strength info | | |
|---|---|---|---|---|---|---|---|---|---|
| | Nb. iter | Time (s) | Nb. iter | Time (s) | Nb. iter | Time | Nb. strength | Max strength | Avg strength |
| c100_400_10_F_L_10 | 897 | 106.42 | 902 | 105.5 | 0.56 | −0.86 | 56 | 21.88 | 9.84 |
| c100_400_10_F_T_10 | 2170 | 491.34 | 2083 | 495.85 | −4.01 | 0.92 | 59 | 33.49 | 11.62 |
| c100_400_10_V_L_10 | 1361 | 312.08 | 1352 | 254.95 | −0.66 | −18.31 | 88 | 22.22 | 9.40 |
| c100_400_30_F_L_10 | 6109 | 10871.3 | 5240 | 9678.83 | −14.22 | −10.97 | 1620 | 40.20 | 5.19 |
| c100_400_30_F_T_10 | 3374 | 2361.29 | 3027 | 2085 | −10.28 | −11.70 | 1070 | 41.78 | 7.34 |
| c25_100_10_F_L_5 | 171 | 1.06 | 158 | 1.09 | −7.60 | 2.83 | 31 | 36.84 | 12.72 |
| c25_100_10_F_T_5 | 133 | 0.71 | 127 | 0.73 | −4.51 | 2.82 | 14 | 25.68 | 14.11 |
| c25_100_10_V_L_5 | 115 | 0.64 | 95 | 0.54 | −17.39 | −15.63 | 25 | 25.81 | 17.07 |
| c25_100_30_F_L_5 | 300 | 6.21 | 233 | 5.09 | −22.33 | −18.04 | 105 | 45.16 | 13.13 |
| c25_100_30_F_T_5 | 226 | 3.7 | 164 | 2.86 | −27.43 | −22.70 | 99 | 69.80 | 22.32 |
| c25_100_30_V_T_5 | 233 | 4.04 | 182 | 3.22 | −21.89 | −20.30 | 118 | 51.31 | 15.21 |
| Mean | | | | | −11.80 | −10.18 | 298.63 | 37.65 | 12.54 |

**Table 8** Effect of using Benders inequalities associated to extreme rays

| Instance | Benders inequalities | | Benders inequalities (extreme rays) | |
|---|---|---|---|---|
| | Nb. iter | Time (s) | Nb. iter | Time (s) |
| R01 | 40.33 | 0.14 | **34.67** | **0.13** |
| R02 | 44.17 | 0.24 | **40.00** | 0.24 |
| R03 | 43.67 | 0.39 | **37.00** | **0.34** |
| R04 | **32.44** | **0.15** | 32.89 | 0.17 |
| R05 | **67.33** | **0.65** | 71.22 | 0.81 |
| R06 | 72.33 | 1.32 | **64.56** | **1.26** |
| R07 | 34.00 | **0.19** | **33.56** | 0.24 |
| R08 | 81.89 | 1.15 | **73.56** | 1.15 |
| R09 | 98.44 | 2.74 | **87.44** | **2.70** |
| R10 | 225.11 | **11.59** | **202.56** | 14.58 |
| R11 | 205.56 | **37.95** | **154.11** | 38.10 |
| R12 | 143.22 | 38.00 | **116.89** | **33.45** |
| R13 | 396.33 | 107.65 | **340.78** | **89.32** |
| R14 | **581.33** | **441.89** | 694.11 | 782.58 |
| R15 | **347.00** | **492.39** | 414.22 | 1760.19 |
| R16 | 347.11 | 158.10 | **325.33** | **126.93** |
| R17 | **963.44** | **1938.99** | 1413.00 | 5348.33 |
| R18 | **638.78** | **2530.32** | 820.11 | 9646.16 |

the strengthening procedure to generate metric inequalities can be used in any algorithm (for the MCFND or for any other network design problem) that alternates between solving a relaxation, gradually improved by the addition of valid inequalities, and restricted multicommodity flow subproblems. Evaluating the performance of the strengthening procedure in this context constitutes an interesting avenue for future research.

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