

Limits and continuity

Definitions

Limits
Let f be a real-valued function.
We say that f has the limit L as x approaches a if:
 $\lim_{x \rightarrow a} f(x) = L$
If f gets arbitrarily close to L whenever x is close enough to a but $x \neq a$.
In other words, the limit L must be a unique real number.

You can define right and left limits
Theorem:
 $\lim_{x \rightarrow a} f(x) = L$ if and only if $\lim_{x \rightarrow a^+} f(x) = L$ and $\lim_{x \rightarrow a^-} f(x) = L$.
Thus the limit exists if and only if the left and right hand limits exist and are equal.
 $\lim_{x \rightarrow a^+} f(x) = L$
 $\lim_{x \rightarrow a^-} f(x) = L$

It also works for x approaching infinity
Limits as x Approaches Infinity
We say that f has the limit L as x approaches positive infinity, $\lim_{x \rightarrow \infty} f(x) = L$, if f gets arbitrarily close to L whenever x is sufficiently large and positive.
We say that f has the limit L as x approaches negative infinity, $\lim_{x \rightarrow -\infty} f(x) = L$, if f gets arbitrarily close to L whenever x is sufficiently large and negative.
Note:
1. L and a must be finite.
2. Limit laws still apply.

Formal definition of limits
A sequence is defined to be a list of real numbers a_1, a_2, a_3, \dots .
A limit L is thought of as an ordered list of real numbers $\epsilon_1, \epsilon_2, \epsilon_3, \dots$.
The sequence is bounded by L within ϵ_n of the n^{th} term.
Example:
 $a_n = 1/n^2$
 $L = 0$
Then, $\lim_{n \rightarrow \infty} a_n = 0$.
Theorem:
Let $\{a_n\}$ be a sequence of real numbers. Then $\lim_{n \rightarrow \infty} a_n = L$ if and only if for every $\epsilon > 0$, there exists a positive integer N such that for all $n > N$, $|a_n - L| < \epsilon$.
Proof:
Let $\epsilon > 0$ be given. We want to find N such that $|a_n - L| < \epsilon$ for all $n > N$.
Since $a_n = 1/n^2$, we have $|a_n - L| = |1/n^2 - 0| = 1/n^2$.
We want $1/n^2 < \epsilon$, which is equivalent to $n^2 > 1/\epsilon$, or $n > 1/\sqrt{\epsilon}$.
Choose $N = \lceil 1/\sqrt{\epsilon} \rceil$. Then for all $n > N$, $n > 1/\sqrt{\epsilon}$, so $n^2 > 1/\epsilon$, and $1/n^2 < \epsilon$.
Therefore, $\lim_{n \rightarrow \infty} 1/n^2 = 0$.

Continuity
Let f be a real-valued function.
The function f is continuous at a if:
 $\lim_{x \rightarrow a} f(x) = f(a)$

Attention for functions that are not defined for the whole set of real numbers.
At the endpoints of a domain, we cannot take both left and right hand limits, so we use the appropriate limit continuity.
1. A function f is continuous from the left at a if $\lim_{x \rightarrow a^-} f(x) = f(a)$.
2. A function f is continuous from the right at a if $\lim_{x \rightarrow a^+} f(x) = f(a)$.

Differentiability
Let f be a real-valued function. The derivative of f at a is defined by:
 $f'(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$
The function f is differentiable at a if the limit exists.
Geometrically, the derivative $f'(a)$ is the gradient of the tangent line to the graph of f at $(a, f(a))$.
Note:
We can also define left derivatives and right derivatives.

Limit laws (1-5 also valid for x approaching infinity)
Let $\lim_{x \rightarrow a} f(x) = L$ and $\lim_{x \rightarrow a} g(x) = M$.
1. $\lim_{x \rightarrow a} (f(x) + g(x)) = L + M$
2. $\lim_{x \rightarrow a} (f(x) - g(x)) = L - M$
3. $\lim_{x \rightarrow a} (c \cdot f(x)) = c \cdot L$
4. $\lim_{x \rightarrow a} (f(x) \cdot g(x)) = L \cdot M$
5. $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{L}{M}$ provided $M \neq 0$
6. $\lim_{x \rightarrow a} \sqrt[n]{f(x)} = \sqrt[n]{L}$ provided $L \geq 0$ if n is even, and $L \in \mathbb{R}$ if n is odd.

Sandwich theorem
Sandwich Theorem:
If $f(x) \leq g(x) \leq h(x)$ when x is near a , and $\lim_{x \rightarrow a} f(x) = L$ and $\lim_{x \rightarrow a} h(x) = L$, then $\lim_{x \rightarrow a} g(x) = L$.

L'Hôpital
L'Hôpital's Rule:
Let f and g be differentiable functions near a , and $f'(a) \neq 0$ at all points x near a , $x \neq a$.
If $\lim_{x \rightarrow a} \frac{f(x)}{g(x)}$ is an indeterminate form $\frac{0}{0}$ or $\frac{\infty}{\infty}$, then:
 $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$
Note:
The limit involving the derivative exists.

Moving the limit inside or outside
Theorem:
If f is continuous at a and $\lim_{x \rightarrow a} g(x) = L$, then:
 $\lim_{x \rightarrow a} f(g(x)) = f(L)$
Note:
The theorem also holds for limits as $x \rightarrow \infty$ or $x \rightarrow -\infty$ as long as f is continuous.

Rules/theorems/tricks

List of some continuous functions
Continuity Theorem 3:
The following functions are continuous at every point in their domains: polynomials, exponential functions, logarithmic functions, trigonometric functions, inverse trigonometric functions.

Composition of continuous functions are continuous in their domains
Let f and g be real-valued functions and let $a \in \mathbb{R}$ be a constant.
Continuity Theorem 1:
If the functions f and g are continuous at a , then the following functions are continuous at a :
1. $f \circ g$
2. $f \cdot g$
3. f/g provided $g(a) \neq 0$.
Note:
The theorem follows from limit laws.
Remember that $(f \circ g)(x) = f(g(x))$.
Continuity Theorem 2:
If f is continuous at a and g is continuous at $f(a)$, then $f \circ g$ is continuous at a .

Be careful with the domains.
Screenshot of a graph showing a function with a jump discontinuity at $x=0$.
1. Screenshot

Differentiability and continuity of a function
Theorem:
If f is differentiable at a , then f is continuous at a .

Definitions

Sequences...
A sequence is defined to be a list of real numbers a_1, a_2, a_3, \dots .
A limit L is thought of as an ordered list of real numbers $\epsilon_1, \epsilon_2, \epsilon_3, \dots$.
The sequence is bounded by L within ϵ_n of the n^{th} term.
Example:
 $a_n = 1/n^2$
 $L = 0$

...and their limits
Limits of Sequences
A sequence $\{a_n\}$ has the limit L if a_n approaches L as n approaches infinity. Note that L must be finite.
The limit:
 $\lim_{n \rightarrow \infty} a_n = L$
If L is not finite, we say that the sequence diverges. Otherwise, we say that the sequence converges.

Series
The series with terms a_n is denoted by the sum:
 $\sum_{n=1}^{\infty} a_n$
If $\sum_{n=1}^{\infty} a_n$ converges, we say that the series converges. Otherwise we say that the series diverges.
Example:
The geometric series $\sum_{n=0}^{\infty} r^n$ converges if $|r| < 1$.
The series $\sum_{n=1}^{\infty} 1/n^p$ converges if $p > 1$ and diverges if $p \leq 1$.
The sequence and series both diverge to infinity, so the sum does not exist.

Geometric series
A geometric series has the form:
 $\sum_{n=0}^{\infty} ar^n$
where $a \in \mathbb{R}$ and $r \in \mathbb{R}$.
The series converges if $|r| < 1$ and diverges if $|r| \geq 1$.
If $|r| < 1$, we have:
 $\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$

Harmonic p series
A harmonic p series has the form:
 $\sum_{n=1}^{\infty} \frac{1}{n^p}$
The series converges if $p > 1$ and diverges if $p \leq 1$.
Example:
 $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges. $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

Connecting limits of real valued functions and limits of sequences
Theorem:
Let f be a real-valued function and $\{a_n\}$ be a sequence of real numbers such that $a_n \rightarrow L$.
Then:
 $\lim_{x \rightarrow L} f(x) = f(L)$ if and only if $\lim_{n \rightarrow \infty} f(a_n) = f(L)$.
This means that we can use the techniques for evaluating limits of functions to evaluate limits of sequences.

The sandwich theorem also works for sequences
Sandwich Theorem:
Let $\{a_n\}, \{b_n\}$ and $\{c_n\}$ be sequences of real numbers. If $a_n \leq b_n \leq c_n$ for all n and $\lim_{n \rightarrow \infty} a_n = L$ and $\lim_{n \rightarrow \infty} c_n = L$, then:
 $\lim_{n \rightarrow \infty} b_n = L$

Composition of sequences
Theorem:
Let $\{a_n\}$ and $\{b_n\}$ be sequences of real numbers and $c \in \mathbb{R}$ a constant.
If $\lim_{n \rightarrow \infty} a_n = L$ and $\lim_{n \rightarrow \infty} b_n = M$, then:
1. $\lim_{n \rightarrow \infty} (a_n + b_n) = L + M$
2. $\lim_{n \rightarrow \infty} (a_n - b_n) = L - M$
3. $\lim_{n \rightarrow \infty} (c \cdot a_n) = c \cdot L$
4. $\lim_{n \rightarrow \infty} (a_n \cdot b_n) = L \cdot M$ provided $M \neq 0$.

Standard limits for sequences
Standard Limits:
1. $\lim_{n \rightarrow \infty} \frac{1}{n} = 0$
2. $\lim_{n \rightarrow \infty} \frac{1}{n^p} = 0$ for $p > 0$
3. $\lim_{n \rightarrow \infty} n^p = \infty$ for $p > 0$
4. $\lim_{n \rightarrow \infty} \frac{1}{n^p} = 0$ for $p > 0$
5. $\lim_{n \rightarrow \infty} n^p = \infty$ for $p > 0$
6. $\lim_{n \rightarrow \infty} \frac{1}{n^p} = 0$ for $p > 0$
7. $\lim_{n \rightarrow \infty} n^p = \infty$ for $p > 0$
Note:
Standard limits (1), (2), (3), (4), (5), (6), (7) also hold for limits of real-valued functions as $x \rightarrow \infty$. Standard limit (8) also holds for $x \rightarrow -\infty$ when $p > 1$.

Helping you identify sequences hierarchy
Note:
This hierarchy can be used to help identify the largest limit in an expression.
 $\lim_{x \rightarrow \infty} x^2 = \infty$
 $\lim_{x \rightarrow \infty} x = \infty$
 $\lim_{x \rightarrow \infty} \frac{1}{x} = 0$

Properties of series
Properties of Series:
Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be series, and $c \in \mathbb{R}$ a constant.
1. $\sum_{n=1}^{\infty} (a_n + b_n)$ converges if and only if $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converge, and then $\sum_{n=1}^{\infty} (a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$.
2. $\sum_{n=1}^{\infty} (c \cdot a_n)$ converges if and only if $\sum_{n=1}^{\infty} a_n$ converges, and then $\sum_{n=1}^{\infty} (c \cdot a_n) = c \cdot \sum_{n=1}^{\infty} a_n$.
3. $\sum_{n=1}^{\infty} a_n$ diverges then $\sum_{n=1}^{\infty} c \cdot a_n$ diverges.
Note:
These follow from the properties of the limits of sequences.

Ratio test
Ratio Test:
Let $\sum_{n=1}^{\infty} a_n$ be a positive term series and $L = \lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n}$.
1. If $L < 1$, $\sum_{n=1}^{\infty} a_n$ converges.
2. If $L > 1$, $\sum_{n=1}^{\infty} a_n$ diverges.
3. If $L = 1$, the test fails to be conclusive.
Note:
The ratio test is useful if f contains an exponential or factorial function.

Convergence and divergence tests for series
Theorem:
If $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n \rightarrow \infty} a_n = 0$.
Note:
The converse is not true. $\lim_{n \rightarrow \infty} a_n = 0$ does not imply that $\sum_{n=1}^{\infty} a_n$ converges.
Some examples:
1. $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.
2. $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.
3. $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$ and diverges if $p \leq 1$.

Divergence test via the limit of the associated sequence
Divergence Test:
If $\lim_{n \rightarrow \infty} a_n \neq 0$ then $\sum_{n=1}^{\infty} a_n$ diverges.
Note:
1. $\lim_{n \rightarrow \infty} a_n = 0$ does not imply convergence or divergence.
2. The Divergence Test is not robust, so we need to use another test to determine if $\sum_{n=1}^{\infty} a_n$ converges or diverges.

Comparison test for positive term series
Comparison Test:
Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be positive term series.
1. If $a_n \leq b_n$ for all n and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.
2. If $a_n \geq b_n$ for all n and $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.
Note:
To apply the comparison test we compare a given series to a known p series or geometric series.

Sequences and series

Rules/theorems/tricks