


Section 2:
Hyperbolic
functions

Definitions


Even functions 85

Even Functions
A function f is an even function if $f(x) = f(-x)$
Example
 $f(x) = \cos x$ and $f(x) = x^2$



Odd functions 85

Odd Functions
A function f is an odd function if $f(x) = -f(-x)$
Example
 $f(x) = \sin x$ and $f(x) = x^3$



Hyperbolic cosine 111 86

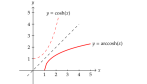
We define the **hyperbolic cosine** function:
 $\cosh x = \frac{1}{2}(e^x + e^{-x}), x \in \mathbb{R}$



Properties

- $\cosh(0) = \frac{1}{2}(1 + 1) = 1$
- $\cosh(-x) = \frac{1}{2}(e^{-x} + e^x) = \cosh x$ (even function).

2. **Inverse hyperbolic cosine function: $\operatorname{arcosh} x$**
Restrict domain of $\cosh x$ to be $[0, \infty)$ to give a 1-1 function. Then:
domain $\operatorname{arcosh} x =$ range $\cosh x = [1, \infty)$
range $\operatorname{arcosh} x =$ restricted domain $\cosh x = [0, \infty)$
 $\cosh(\operatorname{arcosh} x) = x, x \geq 1$
 $\operatorname{arcosh}(\cosh x) = x, x \geq 0$



Hyperbolic sine 111 87

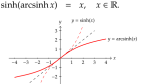
We define the **hyperbolic sine** function:
 $\sinh x = \frac{1}{2}(e^x - e^{-x}), x \in \mathbb{R}$



Properties

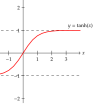
- $\sinh(0) = \frac{1}{2}(1 - 1) = 0$
- $\sinh(-x) = \frac{1}{2}(e^{-x} - e^x) = -\sinh x$ (odd function).

1. **Inverse hyperbolic sine function: $\operatorname{arsinh} x$**
Since $\sinh x$ is a 1-1 function:
domain $\operatorname{arsinh} x =$ range $\sinh x = \mathbb{R}$
range $\operatorname{arsinh} x =$ domain $\sinh x = \mathbb{R}$
 $\operatorname{arsinh}(\sinh x) = x, x \in \mathbb{R}$
 $\sinh(\operatorname{arsinh} x) = x, x \in \mathbb{R}$



Hyperbolic tangent 111 88

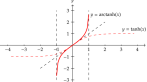
We define the **hyperbolic tangent** function:
 $\tanh x = \frac{\sinh x}{\cosh x} = \frac{\frac{1}{2}(e^x - e^{-x})}{\frac{1}{2}(e^x + e^{-x})} = \frac{e^x - e^{-x}}{e^x + e^{-x}}, x \in \mathbb{R}$



Properties

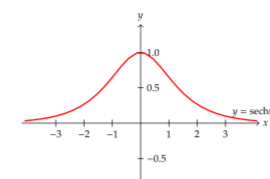
- $\tanh(0) = \frac{\sinh(0)}{\cosh(0)} = \frac{0}{1} = 0$
- $\tanh(-x) = \frac{\sinh(-x)}{\cosh(-x)} = \frac{-\sinh x}{\cosh x} = -\tanh x$ (odd function).

3. **Inverse hyperbolic tangent function: $\operatorname{artanh} x$**
Since $\tanh x$ is a 1-1 function:
domain $\operatorname{artanh} x =$ range $\tanh x = (-1, 1)$
range $\operatorname{artanh} x =$ domain $\tanh x = \mathbb{R}$
 $\operatorname{artanh}(\operatorname{artanh} x) = x, -1 < x < 1$
 $\operatorname{artanh}(\tanh x) = x, x \in \mathbb{R}$



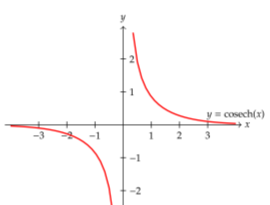
Sech 111 89

$$\operatorname{sech} x = \frac{1}{\cosh x}, x \in \mathbb{R}$$



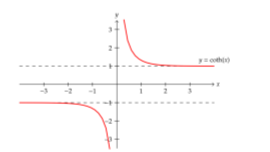
Cosech 111 90

$$\operatorname{cosech} x = \frac{1}{\sinh x}, x \in \mathbb{R} \setminus \{0\}$$



Coth x 111 91

$$\operatorname{coth} x = \frac{1}{\tanh x} = \frac{\cosh x}{\sinh x}, x \in \mathbb{R} \setminus \{0\}$$



Rules/theorems/
tricks

Addition formulae 111 94

Addition Formulae

$$\begin{aligned} \sinh(x+y) &= \sinh x \cosh y + \cosh x \sinh y \\ \cosh(x+y) &= \cosh x \cosh y + \sinh x \sinh y \\ \sinh(x-y) &= \sinh x \cosh y - \cosh x \sinh y \\ \cosh(x-y) &= \cosh x \cosh y - \sinh x \sinh y \end{aligned}$$

Double angle formulae 111 95

Double Angle Formulae

$$\begin{aligned} \sinh(2x) &= 2 \sinh x \cosh x \\ \cosh(2x) &= \cosh^2 x + \sinh^2 x \\ \sinh^2(x) &= \frac{\cosh(2x) - 1}{2} \\ \cosh^2(x) &= \frac{\cosh(2x) + 1}{2} \end{aligned}$$

Basic identities 111 96

Basic Identities

$$\begin{aligned} \cosh^2 x - \sinh^2 x &= 1 \\ \cosh^2 x - 1 &= \sinh^2 x \\ 1 - \sinh^2 x &= \cosh^2 x \end{aligned}$$

Inverse hyperbolic
functions as
derivatives 111 97

The inverse hyperbolic functions can be expressed in terms of natural logarithms.

$$\begin{aligned} \operatorname{arsinh} x &= \log(x + \sqrt{x^2 + 1}), x \in \mathbb{R} \\ \operatorname{arcosh} x &= \log(x + \sqrt{x^2 - 1}), x \geq 1 \\ \operatorname{artanh} x &= \frac{1}{2} \log\left(\frac{1+x}{1-x}\right), -1 < x < 1 \end{aligned}$$

We can also define inverse reciprocal hyperbolic functions:

- $\operatorname{arcsech} x$ ($0 < x \leq 1$)
- $\operatorname{arccosech} x$ ($x \neq 0$)
- $\operatorname{arcoth} x$ ($x < -1$ or $x > 1$)

Derivatives of
hyperbolic
functions 111 98

Derivatives of Hyperbolic Functions

$$\begin{aligned} \frac{d}{dx}(\cosh x) &= \sinh x, x \in \mathbb{R} \\ \frac{d}{dx}(\sinh x) &= \cosh x, x \in \mathbb{R} \\ \frac{d}{dx}(\tanh x) &= \operatorname{sech}^2 x, x \in \mathbb{R} \\ \frac{d}{dx}(\operatorname{sech} x) &= -\operatorname{sech} x \tanh x, x \in \mathbb{R} \\ \frac{d}{dx}(\operatorname{cosech} x) &= -\operatorname{cosech} x \coth x, x \in \mathbb{R} \setminus \{0\} \\ \frac{d}{dx}(\coth x) &= -\operatorname{cosech}^2 x, x \in \mathbb{R} \setminus \{0\} \end{aligned}$$

Derivatives of
inverse hyperbolic
functions 111 99

Derivatives

$$\begin{aligned} \frac{d}{dx}(\operatorname{arsinh} x) &= \frac{1}{\sqrt{x^2 + 1}} (x \in \mathbb{R}) \\ \frac{d}{dx}(\operatorname{arcosh} x) &= \frac{1}{\sqrt{x^2 - 1}} (x > 1) \\ \frac{d}{dx}(\operatorname{artanh} x) &= \frac{1}{1-x^2} (-1 < x < 1) \end{aligned}$$

Each formula is derived using implicit differentiation or by differentiating the logarithm definition of each function.

Derivatives of
inverse hyperbolic
functions 111 100

Derivatives

$$\begin{aligned} \frac{d}{dx}(\operatorname{arcsech} x) &= \frac{1}{\sqrt{1-x^2}} (0 < x \leq 1) \\ \frac{d}{dx}(\operatorname{arccosech} x) &= \frac{-1}{\sqrt{1-x^2}} (x > 0) \\ \frac{d}{dx}(\operatorname{arcoth} x) &= \frac{1}{1-x^2} (x < -1 \text{ or } x > 1) \end{aligned}$$

Each formula is derived using implicit differentiation or by differentiating the logarithm definition of each function.