Section 3: Complex numbers

Definitions

Complex number in terms of the angle with the real axis

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The complex number can be written as

$$\boxed{z \;=\; r(\cos\theta + i\sin\theta)}$$

where

- $r = |z| = \sqrt{x^2 + y^2}$
- $\tan\theta = \dfrac{y}{x}$

Note:

The angle θ is not unique – only defined up to multiples of 2π.
We choose θ such that $-\pi < \theta \leq \pi$ and call this angle the principal argument of z.

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The complex exponential definition and the polar form of a complex number

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The Complex Exponential

We define the complex exponential using Euler's formula

$$\boxed{e^{i\theta} \;=\; \cos\theta + i\sin\theta}$$

for $\theta \in \mathbb{R}$.

We can then write the polar form of a complex number as

$$\boxed{z \;=\; re^{i\theta}}$$

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Properties of the Complex Exponential

1. $\boxed{e^{i0} = 1}$

Proof:
$e^{i0} = \cos 0 + i\sin 0 = 1.$

2. $\boxed{e^{i\theta}e^{i\phi} = e^{i(\theta+\phi)}}$

Proof:

$$\begin{aligned} e^{i\theta}e^{i\phi} &= (\cos\theta + i\sin\theta)\big(\cos\phi + i\sin\phi\big) \\ &= \cos\theta\cos\phi + i\cos\theta\sin\phi + i\sin\theta\cos\phi - \sin\theta\sin\phi \\ &= \big(\cos\theta\cos\phi - \sin\theta\sin\phi\big) + i\big(\cos\theta\sin\phi + \sin\theta\cos\phi\big) \\ &= \cos\big(\theta+\phi\big) + i\sin(\theta+\phi) \\ &= e^{i(\theta+\phi)}. \end{aligned}$$

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Sin and Cos in terms of the complex exponential

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$$\Rightarrow \boxed{\sin\theta = \frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right)}$$

$$\Rightarrow \boxed{\cos\theta = \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right)}$$

These formulae give a connection between the hyperbolic and trigonometric functions.

$$\begin{aligned} \cosh(i\theta) &= \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right) = \cos\theta \\ \sinh(i\theta) &= \frac{1}{2}\left(e^{i\theta} - e^{-i\theta}\right) = i\sin\theta \end{aligned}$$

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Rules/theorems/tricks

Products and division in polar form

Products and Division in Polar Form

If $z = r_1e^{i\theta}$ and $w = r_2e^{i\phi}$ then

$$\boxed{\begin{aligned} zw &= r_1r_2e^{i(\theta+\phi)} \\ \frac{z}{w} &= \frac{r_1}{r_2}e^{i(\theta-\phi)} \end{aligned}}$$

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De Moivre's theorem

De Moivre's Theorem:

If $z = re^{i\theta}$ and n is a positive integer then

$$\boxed{z^n = \left(re^{i\theta}\right)^n = r^ne^{in\theta}}$$

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Differentiation

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Differentiation via the Complex Exponential

If $z = x + yi$ where $x, y \in \mathbb{R}$ then we define

$$e^z = e^{x+iy} = e^x\,e^{iy} = e^x(\cos y + i\sin y).$$

Derivatives of functions from $\mathbb{R}$ to $\mathbb{C}$ are defined similarly as those from $\mathbb{R}$ to $\mathbb{R}$.

Differentiation to functions from $\mathbb{R}$ to $\mathbb{C}$ is also linear and follows the product law.

Integration

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Integration via the Complex Exponential

Since $\frac{d}{dx}\left(e^{kx}\right) = k\,e^{kx}$ if $k = a + bi\ (a, b \in \mathbb{R})$, then

$$\begin{aligned} \int k\,e^{kx}\,dx &= e^{kx} + C \\ \Rightarrow \int e^{kx}\,dx &= \frac{1}{k}e^{kx} + D \end{aligned}$$

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