section 4: Integral calculus

Techniques

Derivative substitutions

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To evaluate

$$\int f[g(x)]g'(x)dx$$

put $u = g(x) \Rightarrow \frac{du}{dx} = g'(x)$.

Then

$$\int f[g(x)]g'(x)dx = \int f(u)\frac{du}{dx}\,dx = \int f(u)\,du$$

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Example 4.1: Evaluate $\int (6x^2 + 10)\sinh(x^3 + 5x - 2)dx$.

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Example 4.2: Evaluate $\int \frac{\operatorname{sech}^2(3x)}{10 + 2\tanh(3x)}dx$.

Trigonometric and Hyperbolic Substitutions

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We can use trigonometric and hyperbolic substitutions to integrate expressions containing

$$\sqrt{a^2 - x^2}, \quad \sqrt{a^2 + x^2}, \quad \sqrt{x^2 - a^2},$$

where a is a positive real number.

Method:

Put $x = g(\theta)$. Then

$$\int f(x)\,dx = \int f[g(\theta)]g'(\theta)\,d\theta$$

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Integrand	Substitution
$\sqrt{a^2 - x^2}$, $\frac{1}{\sqrt{a^2 - x^2}}$, $(a^2 - x^2)^{\frac{3}{2}}$ etc.	$x = a\sin\theta$ or $x = a\cos\theta$
$\sqrt{a^2 + x^2}$, $\frac{1}{\sqrt{a^2 + x^2}}$, $(a^2 + x^2)^{-\frac{3}{2}}$ etc.	$x = a\sinh\theta$
$\sqrt{x^2 - a^2}$, $\frac{1}{\sqrt{x^2 - a^2}}$, $(x^2 - a^2)^{\frac{5}{2}}$ etc.	$x = a\cosh\theta$
$\frac{1}{a^2 + x^2}$	$x = a\tan\theta$

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Example 4.3: Evaluate $\int \frac{1}{\sqrt{x^2 + 25}}dx$ using a substitution.

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Example 4.4: Evaluate $\int \frac{1}{x^2 + 2}dx$ using a substitution.

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Example 4.5: Evaluate $\int \sqrt{9 - 4x^2}\,dx$ if $|x| \le \frac{3}{2}$.

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Example 4.6: Evaluate $\int (x^2 - 1)^{\frac{3}{2}}\,dx$ if $x \ge 1$.

Powers of Hyperbolic Functions

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Consider the integral:

$$\int \sinh^m x \cosh^n x\,dx$$

where m, n are integers (≥ 0).

- If m or n is odd, create a "derivative" substitution by rewriting one of the odd power terms using identities.
- If m and n are even, use double angle formulae.

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Example 4.7: Evaluate $\int \sinh^4\theta\,d\theta$.

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Example 4.8: Evaluate $\int \sinh^5 x \cosh^6 x\,dx$.

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Example 4.9: Evaluate $I = \int \sinh^5 x \cosh^7 x\,dx$.

Partial fractions

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Let $f(x)$ and $g(x)$ be polynomials, then

$$\frac{f(x)}{g(x)} \quad \begin{array}{l} \longrightarrow \text{degree } n \\ \longrightarrow \text{degree } d \end{array}$$

can be written as the sum of partial fractions.

n<d

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Case 1: $n < d$

1. Factorise g over the real numbers.
2. Write down partial fraction expansion.
3. Find unknown coefficients

$$A, A_1, A_2, \ldots, A_r, B, B_1, B_2, \ldots, B_r.$$

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Denominator Factor	Partial Fraction Expansion
$(x - a)$	$\frac{A}{x - a}$
$(x - a)^r$	$\frac{A_1}{x - a} + \frac{A_2}{(x - a)^2} + \cdots + \frac{A_r}{(x - a)^r}$
$(x^2 + bx + c)$	$\frac{Ax + B}{x^2 + bx + c}$
$(x^2 + bx + c)^r$	$\frac{A_1x+B_1}{x^2+bx+c} + \frac{A_2x+B_2}{(x^2+bx+c)^2} + \cdots + \frac{A_rx+B_r}{(x^2+bx+c)^r}$

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Example 4.10: Evaluate $\int \frac{4}{x^2(x + 2)}dx \quad (x \ne 0, -2)$.

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Example 4.11: Evaluate $\int \frac{4x}{(x^2 + 4)(x - 2)}dx \quad (x \ne 2)$.

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Example 4.12: Evaluate $\int \frac{2x^4 + 3x^2}{(x^2 + 1)^2(x^2 + 2)}dx$.

n>d

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Case 2: $n \ge d$

Use long division, then apply case 1.

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Example 4.13: Find

$$\int \frac{5x^4 + 13x^3 + 6x^2 + 4}{x^3 + 2x^2}dx \quad (x \ne 0, -2).$$

Integration by parts

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The product rule for differentiation is

$$\frac{d}{dx}(uv) = \frac{du}{dx}v + u\frac{dv}{dx}$$

Integrate

$$\int \frac{d}{dx}(uv)\,dx = \int \left(\frac{du}{dx}v + u\frac{dv}{dx}\right)dx$$

$$\Rightarrow uv = \int \frac{du}{dx}v\,dx + \int u\frac{dv}{dx}\,dx$$

$$\Rightarrow \boxed{\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx}$$

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Note:
This technique can also be used to integrate inverse trigonometric functions and inverse hyperbolic functions.

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Example 4.14: Evaluate $\int x^2 \log x\,dx \quad (x > 0)$.

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Example 4.15: Evaluate $\int xe^{5x}\,dx$.

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Example 4.17: Evaluate $\int e^{3x}\sin(2x)\,dx$.