Section 5: first order differential equations

Definitions

ODE

Ordinary Differential Equations

(1) An equation of the form

$$f\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \dots, \frac{d^ny}{dx^n}\right) = 0$$

is an ordinary differential equation (o.d.e) of order n.

Separable ODEs

Separable O.D.E'S

A separable first order o.d.e has the form:

$$\frac{dy}{dx} = \mathcal{M}(x)\mathcal{N}(y), \quad (\mathcal{M}(x) \neq 0, \ \mathcal{N}(y) \neq 0)$$

Homogeneous type ODEs

- A homogeneous type o.d.e has the form

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right)$$

Substituting $u = \frac{y}{x}$ reduces the o.d.e to a separable o.d.e.

Linear first order ODE

A linear first order o.d.e has the form:

$$\frac{dy}{dx} + \mathcal{P}(x)y = \mathcal{Q}(x)$$

Bernoulli's equation

- Bernoulli's equation has the form

$$\frac{dy}{dx} + P(x)y = Q(x)y^n$$

Substituting $u = y^{1-n}$ reduces the o.d.e to a linear o.d.e.

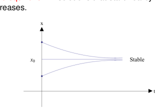
Equilibrium solutions

Equilibrium Solutions

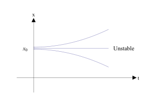
1. An equilibrium solution is a solution that does not change with time.

$$\text{i.e. } \frac{dx}{dt} = 0 \quad \Rightarrow \quad x(t) = x_0$$

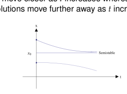
2. Stable equilibrium – solutions that start nearby move closer as t increases.

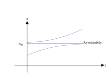


3. Unstable equilibrium – solutions that start nearby move further away as t increases.



4. Semistable equilibrium – on one side of x_0 solutions that start nearby move closer as t increases whereas on the other side of x_0 solutions move further away as t increases.





Transient and steady state terms

Definitions

1. Transient terms: terms decaying to 0 as $t \to \infty$.
2. Steady state terms: terms NOT decaying to 0 as $t \to \infty$.

Rules/theorems/techniques

Solving separable ODEs

Separable O.D.E'S

A separable first order o.d.e has the form:

$$\frac{dy}{dx} = \mathcal{M}(x)\mathcal{N}(y), \quad (\mathcal{M}(x) \neq 0, \ \mathcal{N}(y) \neq 0)$$

To solve use *separation of variables*

$$\begin{aligned} \frac{dy}{dx} &= \mathcal{M}(x)\mathcal{N}(y) \\ \Rightarrow \quad \frac{1}{\mathcal{N}(y)}\frac{dy}{dx} &= \mathcal{M}(x) \\ \Rightarrow \quad \int \frac{1}{\mathcal{N}(y)}\frac{dy}{dx}\,dx &= \int \mathcal{M}(x)\,dx \\ \Rightarrow \quad \int \frac{1}{\mathcal{N}(y)}\,dy &= \int \mathcal{M}(x)\,dx \end{aligned}$$

Separable ODEs

Separable O.D.E'S

A separable first order o.d.e has the form:

$$\frac{dy}{dx} = \mathcal{M}(x)\mathcal{N}(y), \quad (\mathcal{M}(x) \neq 0, \ \mathcal{N}(y) \neq 0)$$

Homogeneous type ODEs

- A homogeneous type o.d.e has the form

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right)$$

Substituting $u = \frac{y}{x}$ reduces the o.d.e to a separable o.d.e.

Example 5.5

Example 5.5: Solve $\frac{dy}{dx} = \frac{y}{1+x} \quad (x \neq -1)$.

Example 5.6

Example 5.6: Solve

$$\frac{dy}{dx} = \frac{1}{2y\sqrt{1-x^2}} \quad (-1 < x < 1, \ y \neq 0)$$

if $y(0) = 3$.

Solving first order ODEs

To solve:

Multiply o.d.e by $\mathcal{I}(x)$

$$\mathcal{I}(x)\frac{dy}{dx} + \mathcal{P}(x)\mathcal{I}(x)y = \mathcal{Q}(x)\mathcal{I}(x)$$

If the left side can be written as the derivative of $y(x)\mathcal{I}(x)$, then

$$\frac{d}{dx}[y(x)\mathcal{I}(x)] = \mathcal{Q}(x)\mathcal{I}(x)$$

which can be solved by integrating with respect to x.

So one integrating factor is

$$\boxed{\mathcal{I}(x) = e^{\int \mathcal{P}dx}}$$

Note:
Since we only need one integrating factor $\mathcal{I}$, we can neglect the '$+c$' and modulus signs when calculating $\mathcal{I}$.

Example 5.8

Example 5.8: Find the general solution of

$$\frac{dy}{dx} + \frac{y}{x} = \sin x \quad (x \neq 0).$$

Example 5.9

Example 5.9: Solve $\frac{1}{2}\frac{dy}{dx} - xy = x$ if $y(0) = -3$.

Phase plots

5. Phase plots:

If $\frac{dx}{dt} = f(x)$, a plot of $\frac{dx}{dt}$ as a function of x will give the equilibrium solutions and the behaviour of solutions close by.

Applications

Population models (Malthus)

Population Models

Malthus (Doomsday) model

Rate of growth is proportional to the population p at time t.

$$\begin{aligned} \frac{dp}{dt} &\propto p \\ \Rightarrow \frac{dp}{dt} &= kp \qquad \text{(separable/linear)} \end{aligned}$$

where k is a constant of proportionality representing net births per unit population per unit time.

If the initial population is $p(0) = p_0$, then the solution is

$$p(t) = p_0 e^{kt}$$

Malthus with harvesting

Doomsday model with harvesting.

Remove some of the population at a constant rate.

$$\frac{dp}{dt} = kp - h, \quad h > 0.$$

Example 5.12: $\frac{dp}{dt} = 3p - 2 \quad (k = 3, h = 2)$

Logistic model

Logistic model.

Include "competition" term in Malthus' model since overcrowding, disease, lack of food and natural resources will cause more deaths.

$$\underbrace{\frac{dp}{dt} = kp}_{\text{net birth rate}} - \underbrace{\frac{k}{a}p^2}_{\text{competition term}} = kp\left(1 - \frac{p}{a}\right)$$

where $a > 0$ is the carrying capacity.

Example 5.13

Example 5.13: $\frac{dp}{dt} = p\left(1 - \frac{p}{4}\right) \quad (k = 1, a = 4)$

Logistic model with harvesting

Logistic model with harvesting.
Remove some of the population at constant rate:

$$\frac{dp}{dt} = kp\left(1 - \frac{p}{a}\right) - h, \quad h > 0, \ a > 0$$

Example 5.14

Example 5.14:

$$\frac{dp}{dt} = p\left(1 - \frac{p}{4}\right) - \frac{3}{4} \quad \left(a = 4, \ k = 1, \ h = \frac{3}{4}\right)$$

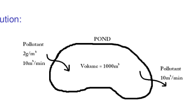
Mixing problems

Mixing Problems

Example 5.15: Effluent (pollutant concentration $2g/m^3$) flows into a pond (volume $1000m^3$, initially $100g$ pollutant) at a rate of $10m^3/min$. The pollutant mixes quickly and uniformly with pond water and flows out of pond at a rate of $10m^3/min$.

Find the concentration of pollutant in the pond at any time.

Solution:



Example 5.16

Example 5.16: Find the concentration of pollutant in pond if input flow rate is decreased to $5m^3/min$.