Section 6 – Second order differential equations

Definitions

Homogeneous/ inhomogeneous second order o.d.e.

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A second order o.d.e has the form

$$F\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}\right) = 0$$

The general form of a linear second order o.d.e is

$$\frac{d^2y}{dx^2} + \mathcal{P}(x)\frac{dy}{dx} + \mathcal{Q}(x)y = \mathcal{R}(x)$$

- If $\mathcal{R}(x) = 0$, the o.d.e is homogeneous (H).
- If $\mathcal{R}(x) \neq 0$, the o.d.e is inhomogeneous (IH).

Note:
A homogeneous linear o.d.e is different to a homogeneous type first order o.d.e.

Initial value and boundary value problems

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Initial value problem for a second order o.d.e

Solve
$$\frac{d^2y}{dx^2} + \mathcal{P}(x)\frac{dy}{dx} + \mathcal{Q}(x)y = \mathcal{R}(x)$$
subject to the conditions $y(x_0) = y_0$ and $y'(x_0) = y_1$.

Boundary value problem for a second order o.d.e

Solve
$$\frac{d^2y}{dx^2} + \mathcal{P}(x)\frac{dy}{dx} + \mathcal{Q}(x)y = \mathcal{R}(x)$$
subject to the conditions $y(a) = y_0$ and $y(b) = y_1$.

Linear independence

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Definition:
Two functions y_1 and y_2 are linearly independent if

$$c_1y_1(x) + c_2y_2(x) = 0 \;\Rightarrow\; c_1 = c_2 = 0.$$

Application/ Modelling

Springs

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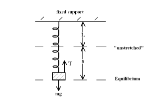
Springs - Free Vibrations

An object (mass m kg) stretches a spring (natural length L m) hanging from a fixed support by s m.



Springs (with forced vibrations / inputs)

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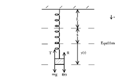
Springs - Forced Vibrations

If an external downwards force f is applied to the spring-mass system at time t, the forces acting on the mass are:



Electrical circuits

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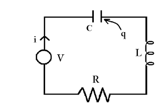
RLC series electric circuit

An RLC series electric circuit is an electric circuit with 4 components connected sequentially in a loop:



Circuits such as this are common in radio communications.

Solution methods

Homogeneous

2nd order ODE with constant coefficients

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Homogeneous 2^{nd} Order Linear O.D.E's with Constant Coefficients

General form:
$$ay'' + by' + cy = 0$$
where a, b, c are constants.

To solve for $y(x)$:

Try $y(x) = e^{\lambda x}$

Distinct real roots

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Case 1: $b^2 - 4ac > 0$

- 2 distinct real values λ_1, λ_2
- 2 linearly independent solutions
$$e^{\lambda_1 x}, \quad e^{\lambda_2 x}$$
- General Solution:
$$\boxed{y(x) = Ae^{\lambda_1 x} + Be^{\lambda_2 x}}$$

One real root

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Case 2: $b^2 - 4ac = 0$

- 1 real value $\lambda = \frac{-b}{2a}$
- 1 solution is $e^{\lambda x}$
- 2^{nd} linearly independent solution is $xe^{\lambda x}$ (found using variation of parameters — not in syllabus).
- General Solution:
$$\boxed{y(x) = Ae^{\lambda x} + Bxe^{\lambda x}}$$

Complex conjugate roots

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Case 3: $b^2 - 4ac < 0$

- 2 complex conjugate values
$$\lambda_1 = \alpha + i\beta, \quad \lambda_2 = \alpha - i\beta$$
- 2 complex linearly independent solutions
$$e^{(\alpha+i\beta)x}, \quad e^{(\alpha-i\beta)x}$$

Note:
Imposing real conditions on the o.d.e will always lead to real coefficients A and B.

- 2 real linearly independent solutions
$$e^{\alpha x}\cos(\beta x), \quad e^{\alpha x}\sin(\beta x)$$

Real General Solution:
$$\boxed{y(x) = Ae^{\alpha x}\cos(\beta x) + Be^{\alpha x}\sin(\beta x)}$$

Inhomogeneous

Strategy:

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Step 1: Find the general solution

Step 2: Find a particular solution

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Example 6.6: Solve $y'' + 2y' - 8y = \mathcal{R}(x)$ where

(a) $\mathcal{R}(x) = 1 - 8x^2$

(b) $\mathcal{R}(x) = e^{3x}$

(c) $\mathcal{R}(x) = 85\cos x$

(d) $\mathcal{R}(x) = 3 - 24x^2 + 7e^{3x}$.

Superposition theorem

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Theorem:
A particular solution of
$$ay'' + by' + cy = c_1\mathcal{R}_1(x) + c_2\mathcal{R}_2(x)$$
is $y_{\mathcal{P}}(x) = c_1y_1(x) + c_2y_2(x)$ where

- $y_1(x)$ is a particular solution of $ay'' + by' + cy = \mathcal{R}_1(x)$,
- $y_2(x)$ is a particular solution of $ay'' + by' + cy = \mathcal{R}_2(x)$,
- a, b, c, c_1, c_2 are constants.