

Section 7: functions of two variables z=f(x,y)

Functions of two variables
A function of two variables is a mapping that assigns a unique real number z to each pair of real numbers (x,y) where (x,y) is called the domain of f. The set of all possible values of z is called the range of f.
Example: f(x,y) = x^2 + y^2 then (0,0) is in the domain.

Cartesian equation of a plane and normal vector to a plane
The Cartesian equation of a plane has the form ax + by + cz = d where a, b, c, d are real constants.
The normal vector to the plane is n = (a, b, c).
The plane passing through point (x0, y0, z0) with a normal vector n = (a, b, c) is given by a(x - x0) + b(y - y0) + c(z - z0) = 0.
That is: ax + by + cz = ax0 + by0 + cz0 = d.
where d = ax0 + by0 + cz0.

Level curves
A contour on the surface z = f(x,y) for which z is constant is called a level curve.
The level curves of a surface z = f(x,y) are level curves. As a contour of the surface.
where z = c is a constant.

Limits...
Let f: D -> R be a real valued function.
We say that the limit of f(x,y) as (x,y) approaches (a,b) is L, written as $\lim_{(x,y) \rightarrow (a,b)} f(x,y) = L$, if for every epsilon > 0, there exists a delta > 0 such that for all (x,y) in D with 0 < sqrt((x-a)^2 + (y-b)^2) < delta, we have |f(x,y) - L| < epsilon.

Graphing
Sketch the surface z = f(x,y). The graph of z = f(x,y) is the set of all points (x,y,z) such that z = f(x,y).
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That is: ax + by + cz = ax0 + by0 + cz0 = d.
where d = ax0 + by0 + cz0.

...and continuity
Let f: D -> R be a real valued function.
f is continuous at (a,b) if $\lim_{(x,y) \rightarrow (a,b)} f(x,y) = f(a,b)$.
Note: The continuity theorem for functions of one variable can be generalized to functions of two variables.

First order partial derivatives
Let f: D -> R be a real valued function. The first order partial derivatives of f at (a,b) are given by $f_x(a,b) = \lim_{h \rightarrow 0} \frac{f(a+h, b) - f(a,b)}{h}$ and $f_y(a,b) = \lim_{h \rightarrow 0} \frac{f(a, b+h) - f(a,b)}{h}$.
Note: The partial derivatives of a function of two variables can be generalized to functions of three variables.

Second order partial derivatives
Let f: D -> R be a real valued function. The second order partial derivatives of f at (a,b) are given by $f_{xx}(a,b) = \lim_{h \rightarrow 0} \frac{f_x(a+h, b) - f_x(a,b)}{h}$, $f_{yy}(a,b) = \lim_{h \rightarrow 0} \frac{f_y(a, b+h) - f_y(a,b)}{h}$, $f_{xy}(a,b) = \lim_{h \rightarrow 0} \frac{f_{xy}(a, b+h) - f_{xy}(a,b)}{h}$, and $f_{yx}(a,b) = \lim_{h \rightarrow 0} \frac{f_{yx}(a, b+h) - f_{yx}(a,b)}{h}$.
Theorem: If the second order partial derivatives of f exist and are continuous then $f_{xy} = f_{yx}$.

Tangent planes and differentiability
Let f: D -> R be a real valued function. The tangent plane to the surface z = f(x,y) at the point (a,b,f(a,b)) is given by $z - f(a,b) = f_x(a,b)(x - a) + f_y(a,b)(y - b)$.
The surface z = f(x,y) is differentiable at (a,b) if the tangent plane exists and is unique.

Directional derivatives...
Let f: D -> R be a real valued function. The directional derivative of f at (a,b) in the direction of the unit vector u = (u1, u2) is given by $D_u f(a,b) = f_x(a,b)u1 + f_y(a,b)u2$.
Note: The directional derivative of a function of two variables can be generalized to functions of three variables.

Local maxima, minima and saddle points
A point (a,b) is a local maximum of f if f(a,b) >= f(x,y) for all (x,y) in a neighborhood of (a,b).
A point (a,b) is a local minimum of f if f(a,b) <= f(x,y) for all (x,y) in a neighborhood of (a,b).
A point (a,b) is a saddle point of f if it is neither a local maximum nor a local minimum.

Stationary points
A stationary point of f is a point (a,b) such that $f_x(a,b) = 0$ and $f_y(a,b) = 0$.
To determine if a stationary point is a local maximum, local minimum, or saddle point, we use the Hessian matrix H = [f_xx, f_xy; f_xy, f_yy].
Local Maximum: H is negative definite.
Local Minimum: H is positive definite.
Saddle Point: H is indefinite.

Partial integration
Let f: D -> R be a continuous function over a domain D of R^2. The partial derivatives of f with respect to x and y are denoted by f_x and f_y .
The partial derivative of f with respect to x is denoted by f_x .
The partial derivative of f with respect to y is denoted by f_y .

Double integrals
Let f: D -> R be a continuous function over a domain D of R^2. The double integral of f over D is denoted by $\iint_D f(x,y) dx dy$.
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Over rectangular domains
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Geometrically
Geometric interpretation of f_x and f_y .
Example 7.7: Let f(x,y) = x^2 + y^2. Find f_x and f_y from first principles.
Example 7.8: Let f(x,y) = 3xy + 2y^2. Find f_x and f_y .
Example 7.9: Let f(x,y) = x^2y + y^2xy. Find f_x and f_y .

Techniques/rules/tricks
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... and the Gradient
Let f: D -> R be a differentiable function. We can define the gradient of f at (a,b) as $\nabla f(a,b) = (f_x(a,b), f_y(a,b))$.
Then the directional derivative of f at (a,b) in the direction of the unit vector u = (u1, u2) is given by $D_u f(a,b) = \nabla f(a,b) \cdot u$.

Second derivative test
Let f: D -> R be a real valued function. The second order partial derivatives of f at (a,b) are given by $f_{xx}(a,b)$, $f_{yy}(a,b)$, $f_{xy}(a,b)$, and $f_{yx}(a,b)$.
The Hessian matrix H = [f_xx, f_xy; f_xy, f_yy] is used to determine if a stationary point is a local maximum, local minimum, or saddle point.

Fubini's theorem
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Sketching
The sketch shows a graph of a function of two variables z = f(x,y).
1. Choose the axes.
2. Plot the curve.
3. Label the axes.
4. Label the curve.
5. Label the surface.

Linear approximations
Let f: D -> R be a real valued function. The linear approximation of f at (a,b) is given by $L(x,y) = f(a,b) + f_x(a,b)(x - a) + f_y(a,b)(y - b)$.
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Chain rule
Let f: D -> R be a real valued function. The chain rule for the partial derivatives of f with respect to x and y is given by $f_x = f_x \frac{dx}{dt} + f_y \frac{dy}{dt}$ and $f_y = f_x \frac{dx}{dt} + f_y \frac{dy}{dt}$.
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Properties of gradient and directional derivatives
The directional derivative of f in the direction of the unit vector u = (u1, u2) is given by $D_u f(a,b) = \nabla f(a,b) \cdot u$.
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Example 7.2: Find the level curves of z = sqrt(x^2 + y^2). Hence sketch the surface and label it.

Example 7.3: Sketch the graph of z = 6x^2 + y^2.

Example 7.4: Sketch the graph of z = sqrt(x^2 + y^2).

Example 7.11: Let z = f(x,y) = x^2 + y^2. Find the linear approximation of f at (1,1) and (2,2). Hence, approximate f(1.1, 1.1) and f(2.1, 2.1).

Example 7.12: Find the linear approximation of f(x,y) = x^2 + y^2 at (1,1). Hence, approximate f(1.1, 1.1) and f(2.1, 2.1).

Example 7.13: Let f(x,y) = x^2 + y^2. Find the directional derivative of f at (1,1) in the direction of the unit vector u = (1/sqrt(2), 1/sqrt(2)).

Example 7.14: Let f(x,y) = x^2 + y^2. Find the directional derivative of f at (1,1) in the direction of the unit vector u = (1/sqrt(2), 1/sqrt(2)).

Example 7.15: Let f(x,y) = x^2 + y^2. Find the directional derivative of f at (1,1) in the direction of the unit vector u = (1/sqrt(2), 1/sqrt(2)).

Example 7.20: Find and classify the stationary points of f(x,y) = x^2 + y^2 + z^2. Hence, sketch the surface and label it.

Example 7.21: Find and classify the stationary points of f(x,y) = x^2 + y^2 + z^2. Hence, sketch the surface and label it.

Example 7.24: Evaluate the double integral of f(x,y) = x^2 + y^2 over the domain D = [0,1] x [0,1].

Example 7.25: Using double integrals, find the volume of the wedge shown below.

Definitions

Derivatives

Local maxima, minima and saddle points

Integration

Techniques/rules/tricks

Properties of gradient and directional derivatives

Second derivative test

Fubini's theorem

Over rectangular domains