On Discrete-Time Partial Averaging

Hengchang Liu, Zeyu Li, Gijo Sebastian, Ying Tan, Denny Oetomo

***Abstract*—The majority of literature for partial averaging techniques for discrete-time systems has focused on showing the local exponential stability properties of nonlinear time-varying systems when the partial averaged system is uniformly locally exponential stable. The work generalizes the existing partial averaging techniques to a more general class of systems in which the partial averaged system is uniformly globally asymptotically stable (UGAS). By using closeness of solutions (Proposition 1) between the original system and the partial averaged system, the main result shows that the original time-varying system will be semi-globally practically asymptotically (SPA) stable provided that the partial averaged systems is UGAS and the tuning parameters are sufficiently small. A few other stability results of the original system from different stability properties of its partial averaged system are also presented. Simulation results support our theoretical findings.**

I. Introduction

Averaging theory is one of the most widely used tools to conclude the stability properties of a class of parameterized nonlinear time-varying systems from stability properties of their approximate time-invariant parameter-free averaged system. This method has been extensively used in the stability analysis of adaptive system [9], [19], extremum seeking control for both deterministic dynamics [20], [23] and stochastic systems [10], [11], pulse width modulations [16], vibrational control [3], [4], fast switching systems [12], [24], and so on.

Partial averaging technique [6] is a special case of averaged techniques, in which the nonlinear dynamics have both slowly time-varying components and fast time-varying components. Partial averaging technique averages the dynamics over the fast time-scale while treats the slow dynamics as slowly varying system [8, Chapter 10]. In [14], when the partial averaged continuous-time system is uniformly globally exponentially stable (UGES), the original continuous-time nonlinear time-varying (NLTV) system is uniformly locally exponentially stable (ULES) when the time-scale parameters are selected sufficiently small. A more general case can be found in [23, Section 3], in which both stability properties and robustness in terms of input-to-state stability were discussed.

Compared with rich literature in continuous-time systems, there is much less literature for partial averaging techniques for discrete-time systems. Discrete-time systems come from discretization of continuous-time dynamics or some natural discrete-time behaviours coming from population models [7], inventory models [5], and so on. In particular, due to prevalence of digital technology, investigating the partial averaging of discrete-time NLTV systems with two time-scales will greatly simplify the stability analysis of such systems. Most partial averaging results for discrete-time system in the literature have focused on local exponential stability properties of the original nonlinear NLTV system and its partial averaged system [2], [17]. For example, in [17], it was shown that if the NLTV partial averaged system is locally exponentially stable and both the original system and its partial averaged system have identical equilibrium points, then the original system is also locally exponentially stable. As the partial averaged system is NLTV, it can exhibit much richer stability properties such as uniform global or local asymptotic stability. Hence, the existing results cannot be directly applicable. Moreover, sometimes, due to the existence of parameter uncertainties, the partial averaged system might not have the same equilibria as the original system. Under such a condition, even if the partial averaged system is uniformly locally exponentially stable (ULES), we cannot conclude that the original system is also ULES. Furthermore, in [2], [17], the nonlinear mappings are assumed to be globally Lipschitz continuous, which is quite restrictive.

This work extends the existing partial averaging techniques to a more general settings, which include a) the weaker stability properties of NLTV partial averaged system; b) weaker continuity conditions for NLTV original system and its partial averaged system; c) weaker condition in terms of equilibria of two systems. Moreover, we also provide two versions of the closeness of solutions between the NLTV original system and its partial averaged system, which can be used to show different stability properties.

The paper is organized as follows. Section 2 provides the needed preliminaries, followed by the problem formulation. Section 3 presents our main results including closeness of solutions and three stability results. In Section 4, two illustrative examples are used to demonstrate how the obtained results work, followed by the summary and future work. The proofs are given in Appendix.

II. Preliminaries and Problem Formulation

Let $\mathcal{N}_{\geq 0}$ be the set containing all non-negative integers. The notation $\mathcal{R}$ represents the set of all real numbers. For any vector $\mathbf{x} \in \mathcal{R}^n$, $|\mathbf{x}|$ represents its Euclidean norm, which is defined as $|\mathbf{x}| \triangleq \sqrt{\mathbf{x}^\intercal \mathbf{x}}$, where $(\cdot)^\intercal$ represents the transpose. A continuous function $\alpha : R_{\geq 0} \to R_{\geq 0}$ is said to be of class $\mathcal{K}$ if it is zero at zero and strictly increasing. A continuous function $\sigma : R_{\geq 0} \to R_{\geq 0}$ is said to be of class $\mathcal{L}$ if it is converging to zero as its argument grows unbounded. A continuous function $\beta : [0, a) \times [0, \infty) \to [0, \infty)$ is said to belong to class $\mathcal{KL}$ if, for each fixed s, the mapping $\beta(r, s)$ belongs to class $\mathcal{K}$ with respect to r and for each fixed r, the mapping $\beta(r, s)$ is decreasing with respect to s and $\beta(r, s) \to 0$ as $s \to \infty$ [8].

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The following family of discrete-time NLTV systems with two time-scales are considered:

$$\Sigma_D: \begin{cases} \mathbf{x}(k+1) &= \mathbf{x}(k)+\varepsilon\mathbf{f}(k,\varepsilon k,\mathbf{x}(k),\boldsymbol{\nu}) \\ \mathbf{x}(k_0) &= \mathbf{x}_0 \in \mathcal{R}^n, k_0 \in \mathcal{N}_{\geq 0} \end{cases}, \quad (1)$$

where $\varepsilon \in (0,\varepsilon_0)$ is a small positive parameter used to form time-scale separation. The variable $\boldsymbol{\nu}$ is a set of parameters as the perturbation. The nonlinear mapping $\mathbf{f}(\cdot,\cdot,\cdot,\cdot): \mathcal{N}_{\geq 0} \times \mathcal{R}_{\geq 0} \times \mathcal{R}^n \times \mathcal{D}_\nu \to \mathcal{R}^n$ satisfies some assumptions where $\mathcal{D}_\nu \subset \mathcal{R}^m$ is a compact set.

Remark 1: Although the form of discrete-time NLTV parameterized system used in (1) is quite general as it appeared frequently in literature, for example, [1], [2], [15], [18], and reference therein, we still can build a more general form compared with the system (1):

$$\Sigma_D^a: \begin{cases} \mathbf{z}(k+1) &= \mathbf{f}^a(k,\varepsilon k,\mathbf{z}(k),\boldsymbol{\nu}), \\ \mathbf{z}(k_0) &= \mathbf{x}_0 \in \mathcal{R}^n, k_0 \in \mathcal{N}_{\geq 0} \end{cases}, \quad (2)$$

which also has two time-scales. We will discuss how to link the obtained results for the system (1) to this more general case (see Remark 7). ○

Remark 2: Currently, the dynamics (1) have two time-scales. One is the fast time-varying component (k) and the other is slowly time-varying component (εk), which is in the same time-scale with the slow dynamics $\mathbf{x}(k)$. We can extend the similar concept to discrete-time dynamics with three time-scales, for example, the dynamical system in the following format:

$$\begin{cases} \mathbf{x}(k+1) &= \mathbf{x}(k)+\varepsilon_1\mathbf{f}(k,\varepsilon_2 k,\mathbf{x}(k),\boldsymbol{\nu}) \\ \mathbf{x}(k_0) &= \mathbf{x}_0 \in \mathcal{R}^n, k_0 \in \mathcal{N}_{\geq 0} \end{cases}. \quad (3)$$

where $\varepsilon_1,\varepsilon_2$ are two small positive parameters. The relationship between $\varepsilon_1,\varepsilon_2$ are specified here. When $\varepsilon_1 \ll \varepsilon_2$, we can treat $\varepsilon_2 k$ as a part of fast time-varying component so that classic averaging technique [2] can be applied. When $\varepsilon_1 \gg \varepsilon_2$, we can treat $\varepsilon_2 k$ as a slowly varying parameter to system (3), then classic averaging technique [2] and slowly varying system analysis [8, Chapter 9] can be applied. When $\varepsilon_1 = O(\varepsilon_2)$, system (3) is consistent with system (1). ○

Remark 3: For system (1), different from continuous-time setting as in [14, Equation (3)]: $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x},t,\alpha t)$, in which $\alpha \gg 1$ is used to ensure the existence of fast time-varying components. For a discrete-time system, the fast time-varying components cannot be arbitrarily fast as in the continuous-time. Hence, system (1) is defined in slow time scale, and a small parameter $\varepsilon \ll 1$ is used to separate the time-scale. ○

The first assumption assumes some continuity properties of the nonlinear mapping $\mathbf{f}$.

Assumption 1: The nonlinear mapping $\mathbf{f}(k,s,\mathbf{x},\boldsymbol{\nu})$ is locally Lipschitz continuous with respect to s, $\mathbf{x}$, $\boldsymbol{\nu}$, uniformly in k and s. Moreover, $\mathbf{f}(k,s,\mathbf{x},\boldsymbol{\nu})$ is piece-wisely continuous in k. Also, $\mathbf{f}(k,s,\mathbf{0},\mathbf{0})$ is uniformly bounded in k and s. □

For partial averaging for continuous system [14], locally Lipschitz continuity is needed to ensure the existence and uniqueness of solutions [8, Chapter 3]. For discrete-time nonlinear systems, the condition for existence of solutions is much weaker than its continuous-time counterpart. And locally Lipschitz continuity in Assumption 1 is used to guarantee the closeness of solutions property. Also, for continuous averaging theory, forward complete [22, Definition 3] is required due to the possible finite time escape property. In discrete time, such condition is not required.

It is noted that the key idea of partial averaging techniques is to average the effect of fast time-varying components in the original system (1), which leads to a slowly time-varying partial averaged system. Next assumption assumes the existence of the partial averaged system.

Assumption 2: There is a function $\mathbf{f}_{pa}(s,\mathbf{x}): \mathcal{R}_{\geq 0} \times \mathcal{R}^n \to \mathcal{R}^n$ locally Lipschitz in $(s,\mathbf{x})$, uniformly in s, such that for any given $\varepsilon \in (0,\varepsilon_0)$, there exist $\mathcal{KL}$ function β_{pa} and $K^* \in \mathcal{N}_{\geq 0}$ such that $\forall k \geq k_0 \geq 0$, $\forall K \geq K^*$, $\forall \mathbf{x} \in \mathcal{R}^n$, the following inequality holds:

$$\left| \mathbf{f}_{pa}(\varepsilon k,\mathbf{x}) - \frac{1}{K}\sum_{i=k}^{k+K-1}\mathbf{f}(i,\varepsilon k,\mathbf{x},\mathbf{0}) \right| \leq \beta_{pa}(\max\{|\mathbf{x}|,1\},K). \quad (4)$$

$\mathbf{f}_{pa}$ is said to be the partial average of $\mathbf{f}$. Moreover, $\mathbf{f}_{pa}(\varepsilon k,\mathbf{0})$ is uniformly bounded in k and ε. □

If Assumption 2 holds, the following family of partial averaged systems is generated:

$$\Sigma_D^p: \begin{cases} \mathbf{x}_p(k+1) &= \mathbf{x}_p(k)+\varepsilon\mathbf{f}_{pa}(\varepsilon k,\mathbf{x}_p(k)) \\ \mathbf{x}_p(k_0) &= \mathbf{x}_0 \in \mathcal{R}^n, k_0 \in \mathcal{N}_{\geq 0} \end{cases}. \quad (5)$$

Remark 4: It is noted that both the discrete-time NLTV system (1) and its partial averaged system (5) are parameterized by the small parameter ε. Investigating the stability of two systems uniformly with ε is challenging, though such uniform stability with respect to ε is relatively easier in continuous-time by changing the time-scale. For example, in the continuous-time averaging, the following two systems are the same system at different time-scale.

$$\dot{\mathbf{x}} = \mathbf{f}_c\left(\frac{t}{\varepsilon},\mathbf{x}\right), \mathbf{x}(t_0) = \mathbf{x}_0 \in \mathcal{R}^n \quad (6)$$

$$\frac{d\mathbf{x}}{d\tau} = \varepsilon\mathbf{f}_c(\tau,\mathbf{x}), \mathbf{x}(\tau_0) = \mathbf{x}_0 \in \mathcal{R}^n, \quad (7)$$

where $\tau = \frac{t}{\varepsilon}$ for some positive ε. The singularity coming from $\varepsilon \to 0$ can be solved by time-scale changes as indicted in [8]. The averaged system introduced in time "t" has the following form:

$$\dot{\mathbf{x}} = \mathbf{f}_{c,a}(\mathbf{x}), \mathbf{x}(0) = \mathbf{x}_0 \in \mathcal{R}^n, \quad (8)$$

where $\mathbf{f}_{c,a}(\cdot): \mathcal{R}^n \to \mathcal{R}^n$. This system is both time-invariant and parameter-invariant. Hence its stability is uniform with the initial time and small positive parameter ε. By using the closeness of solutions between the system (6) and its averaged system (8), we can conclude the uniform stability of the system (6). In discrete-time, changing time-scale is relatively more difficult than in continuous, hence, the partial averaged system (5) is still parameter-dependent, leading to parameter-dependent stability properties. ○

III. MAIN RESULTS

This section presents a family of stability results to conclude the stability properties of the original system (1) from the stability properties of the partial averaged system (5). The main result (Theorem 1) shows that if the partial averaged system (5) is UGAS, uniformly in ε, then the original system (1) is semi-globally practically asymptotically (SPA) stable, uniformly in ε. A few corollaries are presented as an extension of Theorem 1. This section starts from the closeness of solutions in a finite time interval between original system (1) and its partial averaged system (5).

A. Closeness of solutions on a finite time interval

The closeness of solutions in a finite time interval plays a crucial role to prove the stability results. Moreover, these results are useful on its own since they characterize the approximating properties of the partial averaged system.

We consider a finite interval $k \in [k_0, k_0 + [\frac{N}{\epsilon}]]$[1], for any $N \in \mathcal{N}_{\geq 0}$. We divide the interval into sub-intervals of the form $[k_l, k_{l+1}]$ where l is the element of the index set $I_\varepsilon = \{0, \ldots, \left[\frac{N}{\varepsilon h_\varepsilon}\right]\}$. Except for the last one, each sub-interval has same length h_ε which is a function of ε satisfying $\lim\limits_{\varepsilon \to 0} h_\varepsilon = \infty$ and $\lim\limits_{\varepsilon \to 0} \varepsilon h_\epsilon = 0$. With initial value $\mathbf{x}(k_0) = \mathbf{x}_p(k_0) = \mathbf{x}_0 \in \mathcal{R}^n$, we can write the solutions of the original system (1) and its partial averaged system (5) for the interval $k \in [k_l, k_{l+1}]$ respectively:

$$\mathbf{x}(k) = \mathbf{x}(k_l) + \varepsilon \sum_{i=k_l}^{k-1} \mathbf{f}(i, \varepsilon i, \mathbf{x}(i), \boldsymbol{\nu}), \tag{9}$$

$$\mathbf{x}_p(k) = \mathbf{x}_p(k_l) + \varepsilon \sum_{i=k_l}^{k-1} \mathbf{f}_{pa}(\varepsilon i, \mathbf{x}_p(i)). \tag{10}$$

To prove the closeness of solutions, two auxiliary series $\boldsymbol{\xi}(k)$ and $\boldsymbol{\omega}(k)$ are used:

$$\boldsymbol{\xi}(k) = \boldsymbol{\xi}(k_l) + \varepsilon \sum_{i=k_l}^{k-1} \mathbf{f}(i, \varepsilon k_l, \boldsymbol{\xi}(k_l)), \tag{11}$$

$$\boldsymbol{\omega}(k) = \boldsymbol{\omega}(k_l) + \varepsilon \sum_{i=k_l}^{k-1} \mathbf{f}_{pa}(\varepsilon k_l, \boldsymbol{\xi}(k_l)), \tag{12}$$

where $\boldsymbol{\xi}(0) = \boldsymbol{\omega}(0) = \mathbf{x}_0$, $\boldsymbol{\omega}(k_l) = \mathbf{x}_p(k_l)$. The result for closeness of solutions in finite time is presented as follows.

Proposition 1: (Closeness of solutions in a finite time interval) Suppose that Assumptions 1, 2 hold. For any positive pair (r, δ) and any positive integer N, there exists a positive pair (ε^*, ν^*) such that for any $\varepsilon \in (0, \varepsilon^*)$, $|\boldsymbol{\nu}| < \nu^*$, any integer $k \in [k_0, k_0 + \left[\frac{N}{\varepsilon}\right]]$, the solutions of the original system (1) and the solutions of the partial averaged system (5) satisfy

$$|\mathbf{x}(k) - \mathbf{x}_p(k)| \leq \delta, \tag{13}$$

for all $|\mathbf{x}(k_0)| = |\mathbf{x}_p(k_0)| = |\mathbf{x}_0| \leq r$. ■

The proof of the above Proposition is provided in Appendix.

[1] $[\frac{N}{\varepsilon}]$ denotes the largest integer i such that $i \leq \frac{N}{\epsilon}$

Compared with Proposition 1, if the original system (1) is not dependent on the parameter $\boldsymbol{\nu}$ and Assumption 3 holds, a stronger closeness of solutions can be obtained.

Assumption 3: There is a function $\mathbf{f}_{pa}(s, \mathbf{x}) : \mathcal{R}_{\geq 0} \times \mathcal{R}^n \to \mathcal{R}^n$ locally Lipschitz in $(s, \mathbf{x})$, uniformly in s, such that for any given $\varepsilon \in (0, \varepsilon_0)$, there exists $\mathcal{L}$ function σ and $K^* \in \mathcal{N}_{\geq 0}$ such that $\forall k \geq k_0 \geq 0$, $\forall K \geq K^*$, $\forall x \in \mathcal{R}^n$, the following inequality holds

$$\left| \mathbf{f}_{pa}(\varepsilon k, \mathbf{x}) - \frac{1}{K} \sum_{i=k}^{k+K-1} \mathbf{f}(i, \varepsilon k, \mathbf{x}, \mathbf{0}) \right| \leq |\mathbf{x}|\, \sigma(K). \tag{14}$$

Moreover $\mathbf{f}(i, \varepsilon k, \mathbf{x}, \mathbf{0}) = \mathbf{f}_{pa}(\varepsilon k, \mathbf{0}) = 0$. □

Remark 5: Compared with the Assumption 2, the Assumption 3 is a stronger assumption. The Assumption 3 shows that when the system (1) and its partial averaged system have the same equilibrium at the origin while the Assumption 2 does not give any assumption about the equilibrium of the two systems. This makes it possible to conclude a stronger closeness of solutions. ◦

Proposition 2: (Strong closeness of solutions in a finite time interval) Let $\boldsymbol{\nu} = 0$ in (1). Suppose that Assumptions 1 and 3 hold. For any positive pair (r, δ) and any positive integer N, there exists a positive ε^* such that for any $\varepsilon \in (0, \varepsilon^*)$, any integer $k \in \left[k_0, k_0 + \left[\frac{N}{\varepsilon}\right]\right]$, the solutions of the original system (1) and the solutions of the partial averaged system (5) satisfy

$$|\mathbf{x}(k) - \mathbf{x}_p(k)| \leq |\mathbf{x}_0|\, \delta, \tag{15}$$

for all $|\mathbf{x}(k_0)| = |\mathbf{x}_p(k_0)| = |\mathbf{x}_0| \leq r$. ■

Due to space limitation, the proof of Proposition 2 is omitted. The result can be obtained by following the similar steps used in the proof of Proposition 1.

B. Stability results

This subsection presents several stability results for the system (1) from the stability properties of its partial averaged system (5). Both systems are time-varying and ε dependent, though (1) also depends on $\boldsymbol{\nu}$. For simplicity, we define UGAS, uniformly in ε, semi-globally exponentially stability, uniformly in ε for the partial averaged system (5). From stability properties of the partial averaged system, we can conclude the stability properties of the original system.

Definition 1: System (5) is UGAS, uniformly in ε if there is a $\mathcal{KL}$ function $\beta(\cdot,\cdot)$ such that for any $\varepsilon \in (0, \varepsilon_0)$, $k \geq k_0 \geq 0$, it follows

$$|\mathbf{x}_p(k)| \leq \beta(|\mathbf{x}_p(k_0)|, (k - k_0)\varepsilon), \tag{16}$$

for all $\mathbf{x}_p(k_0) \in \mathcal{R}^n$. ◦

In Definition 1, though the convergence speed of $\mathbf{x}_p(k)$ depends on ε, the $\mathcal{KL}$ function β is independent of ε. Following the similar definition [21, Definition 1], we say such a stability property is uniformly in ε. The definition of UGES, uniformly in ε is similar as the Definition 1.

Definition 2: System (5) is semi-globally exponential stability, uniformly in ε if there exists $\lambda, A > 0$ such that for any positive r, there exists a positive ε^* such that for any $\varepsilon \in (0, \varepsilon^*)$, $k \geq k_0 \geq 0$, it follows

$$|\mathbf{x}_p(k)| \leq A\,|\mathbf{x}_p(k_0)|\,e^{-\lambda(k-k_0)\varepsilon}, \tag{17}$$

for all $|\mathbf{x}_p(k_0)| \leq r$. ◦

With the closeness of solutions in finite time (Proposition 1) and the assumption that the partial averaged system (5) is UGAS, uniformly in ε, we can conclude a weaker stability properties of the original system (1) as Theorem 1.

Theorem 1: Suppose that Assumptions 1 and 2 hold and the partial averaged system (5) is UGAS, uniformly in ε, with a $\mathcal{KL}$ function β, for any given positive pair (r, δ), there exists positive ε^*, ν^*, such that for any $\varepsilon \in (0, \varepsilon^*)$, $|\boldsymbol{\nu}| < \nu^*$, and $k \geq k_0 \geq 0$, the solutions of (1) satisfy

$$|\mathbf{x}(k)| \leq \max\{\beta(|\mathbf{x}(k_0)|, (k-k_0)\varepsilon), \delta)\}, \tag{18}$$

for all $|\mathbf{x}(k_0)| \leq r$. ■

A sketch of proof of Theorem 1 is presented in Appendix. Theorem 1 shows the original system (1) is SPA (semi-global practically asymptotically) stable, uniformly in ε, when the partial averaged system (5) is UGAS, uniformly in ε. Corollaries 1 and 2 show the stability properties of the original system (1) when its partial averaged system is UGES, uniformly in ε under different types of closeness of solutions.

Corollary 1: Suppose that Assumptions 1 and 2, and the partial averaged system (5) is UGES, uniformly in ε, with a positive pair (A, λ). For any positive pair (r, δ), there exists positive ε^*, ν^*, such that for any $\varepsilon \in (0, \varepsilon^*)$, $|\boldsymbol{\nu}| < \nu^*$, and $k \geq k_0 \geq 0$, the solutions of (1) satisfy

$$|\mathbf{x}(k)| \leq A\,|\mathbf{x}(k_0)|\,e^{-\lambda(k-k_0)\varepsilon} + \delta, \tag{19}$$

for all $|\mathbf{x}(k_0)| \leq r$.

When the strong closeness of solutions (Proposition 2) holds, the original system (1) will have a stronger stability property.

Corollary 2: Let $\boldsymbol{\nu} = 0$ in (1). Suppose that Assumption 1, 3 hold, and the partial averaged system (5) is UGES, uniformly in ε, then the system (1) is semi-globally exponentially stable, uniformly and ε.

Remark 6: In this work, only global stability properties are considered. The idea can be easily extended to local stability properties. The local version of Corollary 2 is consistent with local exponential stability results obtained in [2], [17]. ◦

Remark 7: When a more general class of discrete-time NLTV systems with two time-scales (2) is considered, the concept of multi-step consistency [13, Definition 2] can be adapted to link system (1) with system (2). If two systems have multi-step consistency, they have some closeness of solutions in a finite time interval as shown in [13, Definition 2]. Consequently, the stability results obtained for (1) can be easily extended to (2) as the proof of Theorem 1 indicates. ◦

IV. Simulation Examples

Two simulation examples are presented to demonstrate how partial averaging techniques can work. The first example shows that the solutions of a discrete-time NLTV system with two time scales are closed to the solutions of a partial averaged system in a finite time interval, which verifies Proposition 1. The second example shows that when a partial averaged system is UGES, uniformly in ε, under a stronger assumption, the original system is semi-globally exponentially stable, uniformly in ε as stated in Corollary 2.

A. Example 1

Let us consider the least mean square (LMS) algorithm with time varying parameters. For a linear mapping $y = \mathbf{x}^T\boldsymbol{\omega}_k$, where $\mathbf{x} \in \mathcal{R}^n$, $y \in \mathcal{R}$ are the input and output respectively, and $\boldsymbol{\omega}_k \in \mathcal{R}^n$ is the unknown time varying parameter that we want to estimate using LMS algorithms. It is assumed that the parameters $\boldsymbol{\omega}_k = \boldsymbol{\omega}(\varepsilon k)$ are slowly varying and parametrized by ε. Also the input $\mathbf{x}(k)$ is periodic. The following LMS algorithm can be used [17, Chapter 4]:

$$\hat{\boldsymbol{\omega}}(k+1) = \hat{\boldsymbol{\omega}}(k) - \varepsilon\mathbf{x}(k)\mathbf{x}(k)^T(\hat{\boldsymbol{\omega}}(k) - \boldsymbol{\omega}(\varepsilon k)), \tag{20}$$

where $\hat{\boldsymbol{\omega}}(k)$ are the estimated parameters, and ε is the learning rate. We can see that system (20) has a form of system (1) provided that $\mathbf{x}(k)$ is periodic, thus the discrete-time partial averaging method can be applied to simplify the analysis of the original system system (20), which has the following partial averaged system

$$\hat{\boldsymbol{\omega}}_p(k+1) = \hat{\boldsymbol{\omega}}_p(k) - \varepsilon R(\hat{\boldsymbol{\omega}}_p(k) - \boldsymbol{\omega}(\varepsilon k)), \tag{21}$$

where $R = \lim_{K\to\infty} \frac{1}{K}\sum_{i=k}^{k+K-1} \mathbf{x}(k)\mathbf{x}(k)^T$. Thus our results can be directly applicable.

Next we simulate a scalar case: the input $x(k) = \sin(\frac{\pi}{5}k)$, the unknown parameter $\omega(k) = \cos(\frac{\pi}{5}\varepsilon k)$, $\hat{\omega}(0) = \hat{\omega}_p(0) = 1$. Consequently, the following original system and partial averaged system are obtained:

$$\hat{\omega}(k+1) = \hat{\omega}(k) - \varepsilon\sin^2(\frac{\pi}{5}k)(\hat{\omega}(k) - \cos(\frac{\pi}{5}\varepsilon k)), \tag{22}$$

$$\hat{\omega}_p(k+1) = \hat{\omega}_p(k) - 0.5\varepsilon(\hat{\omega}_p(k) - \cos(\frac{\pi}{5}\varepsilon k)). \tag{23}$$

It is easy to check that Assumptions 1 and 2 satisfy for the system (22) and the system (23). We can see that system (23) is only uniformly bounded. Hence, Proposition 1 is applicable with $\boldsymbol{\nu} = \mathbf{0}$. To verify the results in Proposition 1, $\varepsilon = 0.2$ and $\varepsilon = 0.05$ are selected. It is worthwhile to highlight that when $\varepsilon = 0.05$ is selected, $\hat{\omega}_k$ updates slowly, requiring a large time interval. As Proposition 1 indicates, a smaller ε will lead to a smaller difference between two solutions as shown in Fig.1.

B. Example 2

To show how Corollary 2 works, a simple scalar system is considered:

$$x(k+1) = x(k) - \varepsilon(2 + \sin(\frac{\pi}{4}\varepsilon k))(1 - 2\sin(\frac{\pi}{4}k))x(k), \tag{24}$$

where $x \in \mathcal{R}$ and $k \geq k_0 = 0$. This system has the following partial averaged system:

$$x(k+1) = x(k) - \varepsilon(2 + \sin(\frac{\pi}{4}\varepsilon k))x(k). \tag{25}$$

By Lyapunov method, we can check that the system (25) is UGES, for any $\varepsilon \in (0, \frac{2}{3})$. It is also verified that the system

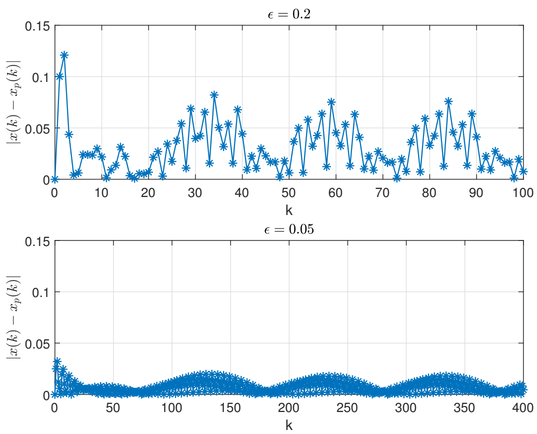

Fig. 1. Distance between x and x_p with $\varepsilon = 0.2$ and $\varepsilon = 0.05$

(24), (25) satisfy Assumptions 1 and 3. As $\nu = \mathbf{0}$, Corollary 2 is thus applicable, indicating that the system (24) is semi-globally exponentially stable, uniformly in ε. Both $\varepsilon = 0.1$ and $\varepsilon = 0.05$ are used with $x(0) = x_p(0) = 7$. Fig.2 shows the trajectories of two systems with different choice of ε, which is consistent with Corollary 2.

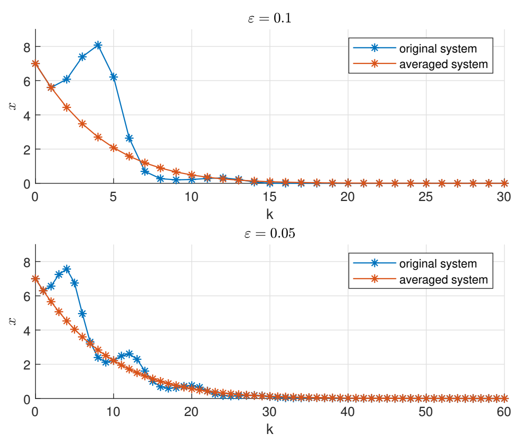

Fig. 2. x and x_p with $\varepsilon = 0.1$ and $\varepsilon = 0.05$

V. Conclusion and Future Work

This work focuses on extending the existing results of partial averaging techniques for discrete-time nonlinear time-varying systems to more general settings when the partial averaged system is uniformly globally stable, instead of uniformly locally exponentially stable. Two types of closeness of solutions between the original system and the partial averaged system are presented, leading to various stability properties of the original system coming from different stability properties of its partial averaged system. The obtained results can be used to analyse the well-known LMS estimation algorithm. Our future work will focus on extending the obtained results to more general systems with the consideration of more general stability properties such as input-to-state stability.

VI. Appendix

Proof of Proposition 1.

For the discrete-time original system (1), and its partial averaged system (5), the existence and uniqueness of the solutions for any given finite time interval can be guaranteed by the local Lipschitz continuity properties of $\mathbf{f}$, $\mathbf{f}_p$ respectively for bounded $\mathbf{x}_0$. Therefore, it is assumed that there exists $R > r$ such that $\{|\mathbf{x}(k)|, |\mathbf{x}_p(k)|\} \leq R$ for any $k \in [k_0, k_0 + [\frac{N}{\varepsilon}]]$.

From Assumptions 1 and 2, there exists a positive constant L, which is the largest Lipschitz constant of $\mathbf{f}$ for $|\mathbf{x}(k)| \leq R$, while L_p is the largest Lipschitz constant for $|\mathbf{x}_p(k)| \leq R$. Also, there exists $P > 0$ such that $\max_{|\mathbf{x}| \leq R}\{|\mathbf{f}(k, \varepsilon k, \mathbf{x}, \mathbf{0})|, |\mathbf{f}_{pa}(\varepsilon k, \mathbf{x})|\} \leq P$.

Next, the difference between two solutions is considered.

$$|\mathbf{x}(k) - \mathbf{x}_p(k)| \leq |\mathbf{x}(k) - \boldsymbol{\xi}(k)| + |\boldsymbol{\xi}(k) - \boldsymbol{\omega}(k)| + |\boldsymbol{\omega}(k) - \mathbf{x}_p(k)|. \tag{26}$$

We can bound three parts in (26) separately.

Step 1 shows the closeness of solutions between $\mathbf{x}(k)$ and $\boldsymbol{\xi}(k)$. For $k \in [k_l, k_{l+1})$, using (9) and (11), it has

$$\begin{aligned}
&|\mathbf{x}(k) - \boldsymbol{\xi}(k)| \\
\leq &|\mathbf{x}(k_l) - \boldsymbol{\xi}(k_l)| + \varepsilon \sum_{i=k_l}^{k-1} |\mathbf{f}(i, \varepsilon i, \mathbf{x}(i), \boldsymbol{\nu}) - \mathbf{f}(i, \varepsilon k_l, \boldsymbol{\xi}(k_l), \mathbf{0})| \\
\leq &|\mathbf{x}(k_l) - \boldsymbol{\xi}(k_l)| + \varepsilon L \sum_{i=k_l}^{k-1} (\varepsilon(i - k_l) + |\mathbf{x}(i) - \boldsymbol{\xi}(k_l)| + |\boldsymbol{\nu}|) \\
\leq &(1 + \varepsilon h_\varepsilon L)|\mathbf{x}(k_l) - \boldsymbol{\xi}(k_l)| + (\varepsilon h_\varepsilon)^2 L(1 + P) + \varepsilon h_\varepsilon L|\boldsymbol{\nu}|.
\end{aligned} \tag{27}$$

Note that $\mathbf{x}(k_0) = \boldsymbol{\xi}(k_0) = \mathbf{x}_0$ and $l \leq \frac{N}{\epsilon h_\epsilon}$, for $k \in [k_0, k_0 + [\frac{N}{\epsilon}]]$, it follows that

$$\begin{aligned}
&|\mathbf{x}(k) - \boldsymbol{\xi}(k)| \\
\leq &((\varepsilon h_\varepsilon)^2 L(1 + P) + \varepsilon h_\varepsilon L|\boldsymbol{\nu}|) \sum_{i=0}^{l-1} (1 + \varepsilon h_\varepsilon L)^i \\
\leq &(\varepsilon h_\varepsilon L(1 + P) + L|\boldsymbol{\nu}|) N e^{NL}.
\end{aligned} \tag{28}$$

For any given positive δ, there exists a positive pair (ε_1^*, ν^*), such that for any $\varepsilon_1 \in (0, \varepsilon_1^*)$ and $|\boldsymbol{\nu}| \leq \nu^*$, $\varepsilon h_\varepsilon L(1 + P)Ne^{NL} \leq \frac{\delta}{6}$ and $L|\boldsymbol{\nu}|Ne^{NL} \leq \frac{\delta}{6}$ holds. This leads to $|\mathbf{x}(k) - \boldsymbol{\xi}(k)| \leq \frac{\delta}{3}$.

Step 2 shows the closeness of solutions between $\boldsymbol{\xi}(k)$ and $\boldsymbol{\omega}(k)$. For $k = k_{l+1}$, by using (11), (12), and Assumption 2, it follows that

$$\begin{aligned}
&|\boldsymbol{\xi}(k_{l+1}) - \boldsymbol{\omega}(k_{l+1})| \\
\leq &|\boldsymbol{\xi}(k_l) - \boldsymbol{\omega}(k_l)| + \varepsilon \sum_{i=k_l}^{k_{l+1}-1} |\mathbf{f}(i, \varepsilon k_l, \boldsymbol{\xi}(k_l), \mathbf{0}) - \mathbf{f}_{pa}(\varepsilon k_l, \boldsymbol{\xi}(k_l))| \\
\leq &|\boldsymbol{\xi}(k_l) - \boldsymbol{\omega}(k_l)| + \varepsilon h_\varepsilon \beta_{pa}(\max\{|\mathbf{x}(k_l)|, 1\}, h_\varepsilon).
\end{aligned} \tag{29}$$

Since $\boldsymbol{\xi}(k_0) = \boldsymbol{\omega}(k_0) = \mathbf{x}_0$, by using induction, for $l \in I_\varepsilon$, the following inequality holds:

$$\begin{aligned}
|\boldsymbol{\xi}(k_l) - \boldsymbol{\omega}(k_l)| \leq &[\frac{N}{\varepsilon h_\varepsilon}]\varepsilon h_\varepsilon \beta_{pa}(\max\{R, 1\}, h_\varepsilon) \\
\leq &N\beta_{pa}(\max\{R, 1\}, h_\varepsilon).
\end{aligned} \tag{30}$$

For $k \in [k_l, k_{l+1})$, using the uniform boundedness of $\mathbf{f}$ and $\mathbf{f}_p$, it yeilds

$$\begin{aligned}&|\boldsymbol{\xi}(k) - \boldsymbol{\omega}(k)| \\ \leq &|\boldsymbol{\xi}(k_l) - \boldsymbol{\omega}(k_l)| + \varepsilon \sum_{i=k_l}^{k-1} |\mathbf{f}(i, \varepsilon k_l, \boldsymbol{\xi}(k_l), \mathbf{0}) - \mathbf{f}_{pa}(\varepsilon k_l, \boldsymbol{\xi}(k_l))| \\ \leq &N\beta_{pa}(\max\{R, 1\}, h_\varepsilon) + 2\varepsilon h_\varepsilon P. \end{aligned} \tag{31}$$

And (31) holds for $k \in [k_0, k_0 + [\frac{N}{\varepsilon}]]$. For any given positive δ, there exists a positive ε_2^* such that for any $\varepsilon_1 \in (0, \varepsilon_2^*)$, it has $|\boldsymbol{\xi}(k) - \boldsymbol{\omega}(k)| \leq \frac{\delta}{3}$.

Step 3 shows the closeness of solutions between $\boldsymbol{\omega}(k)$ and $\mathbf{x}_p(k)$. Using (12) and (10), for $k \in [k_l, k_{l+1})$, we have

$$\begin{aligned}&|\boldsymbol{\omega}(k) - \mathbf{x}_p(k)| \\ \leq &\varepsilon \sum_{i=k_l}^{k-1} |\mathbf{f}_{pa}(\varepsilon k_l, \boldsymbol{\xi}(k_l)) - \mathbf{f}_{pa}(\varepsilon i, \mathbf{x}_p(i))| \\ \leq &\varepsilon L_p \sum_{i=k_l}^{k-1} \varepsilon\,|i - k_l| + |\boldsymbol{\xi}(k_l) - \mathbf{x}_p(i)| \end{aligned} \tag{32}$$

With the help of (12), (31) and discrete Gronwall Lemma, we have

$$\begin{aligned}&|\boldsymbol{\omega}(k) - \mathbf{x}_p(k)| \\ \leq &\varepsilon L_p h_\varepsilon(\varepsilon h_\varepsilon + N\beta_{pa}(\max\{R, 1\}, h_\varepsilon) + 3\varepsilon h_\varepsilon P) \\ &+ \varepsilon L_p \sum_{i=k_l}^{k-1} |\boldsymbol{\omega}(i) - \mathbf{x}_p(i)| \\ \leq &\varepsilon h_\varepsilon L_p(\varepsilon h_\varepsilon(1 + 3P) + N\beta_{pa}(\max\{R, 1\}, h_\varepsilon))e^{\varepsilon h_\varepsilon L_p}. \end{aligned} \tag{33}$$

And (33) holds for $k \in [k_0, k_0 + [\frac{N}{\varepsilon}]]$. For any given positive δ, there exists a positive pair ε_3^* such that for any $\varepsilon_1 \in (0, \varepsilon_3^*)$, the following inequality $|\boldsymbol{\omega}(k) - \mathbf{x}_p(k)| \leq \frac{\delta}{3}$ holds.

Step 4 concludes the result by combining the previous three steps. By selecting $\varepsilon^* = \min\{\varepsilon_1^*, \varepsilon_2^*, \varepsilon_3^*\}$, ν^*, (13) holds. This completes the proof. □

Sketch of Proof of Theorem 1.

If the partial averaged system (5) is UGAS, there exists a $\mathcal{KL}$ function β such that for any given positive pair (r, δ) such that for any $|\mathbf{x}_0| \leq r$, there exists a positive integer $N = N(r, \delta)$ such that $|\mathbf{x}_p(k)| \leq \beta(|\mathbf{x}_0|, (k - k_0)\varepsilon) \leq \frac{\delta}{3}$, for $k = k_0 + [\frac{N}{\varepsilon}]$. For this fixed N, by applying the closeness of solutions on the finite time interval $[k_0, k_0 + [\frac{N}{\varepsilon}]]$ between the original system (1) and its partial averaged system (5), with sufficiently small ε_1, we have $|\mathbf{x}(k) - \mathbf{x}_p(k)| \leq \frac{\delta}{3}$. Now we have $|\mathbf{x}(k)| \leq \frac{2\delta}{3}$ for $k = k_0 + [\frac{N}{\varepsilon}]$.

We will show from the next step, the trajectory will stay in δ-neighborhood of the origin.

Reinitialize $\mathbf{x}_p(k) = \mathbf{x}(k)$ at $k = k_0 + [\frac{N}{\varepsilon}]$, with sufficient small ε_2, the closeness of solutions still holds $|\mathbf{x}(k) - \mathbf{x}_p(k)| \leq \frac{\delta}{3}$, for $k \in [k_0 + [\frac{N}{\varepsilon}], k_0 + 2[\frac{N}{\varepsilon}]]$. This implies $|\mathbf{x}(k)| \leq \delta$ for $k \in [k_0 + [\frac{N}{\varepsilon}], k_0 + 2[\frac{N}{\varepsilon}]]$, and $|\mathbf{x}(k)| \leq \frac{2\delta}{3}$ for $k = k_0 + 2[\frac{N}{\varepsilon}]$.

By induction, with sufficient small ε_2, we have $|\mathbf{x}(k)| \leq \delta$, for $k \geq k_0 + [\frac{N}{\varepsilon}]$. Now with $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$, we have $|\mathbf{x}(k)| \leq \beta(|\mathbf{x}_0|, (k - k_0)\varepsilon) + \frac{\delta}{3}$ for $k \in [k_0, k_0 + [\frac{N}{\varepsilon}]]$, and $|\mathbf{x}(k)| \leq \delta$, for $k \geq k_0 + [\frac{N}{\varepsilon}]$. Therefore, we can conclude that inequality (18) holds.

□

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