On Singular Perturbation for a Class of Discrete-Time Nonlinear Systems in the Presence of Limit Cycles of Fast Dynamics

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Abstract— **This paper extends the existing singular perturbation results to a class of nonlinear discrete-time systems whose fast dynamics have limit cycles. By introducing the discrete-time reduced averaged system, the main result (Theorem 1) shows that for a given fixed time interval, the solutions of the original system can be made arbitrarily close to the solutions of the reduced averaged system and the boundary layer system. From this result, the stability properties of the original system are obtained from the stability properties of the reduced averaged system and the boundary layer system. Simulation results support the theoretical findings.**

I. Introduction

Singular perturbation techniques [1, 13, 12] have been widely used in engineering applications when a dynamic system consists of a fast subsystem and a slow subsystem, see, for example, observer-based feedback control systems [13], the nonlinear high gain observers [12], and extremum seeking control [21], and references therein. The key idea of singular perturbation is time scale separation. More precisely, the slow sub-system can only observe the steady-state behaviors of the fast sub-system while the fast dynamics will treat the slow sub-system as a "constant". This leads to a boundary layer system and a reduced order system, which can be designed independently. Due to its nonlinear nature, the steady-state behaviours of the fast subsystem can have multiple equilibria or multiple limit cycles [12, Chapeter 1], making the stability analysis more complicated for the original system.

There are many existing results for a large class of engineering systems. For continuous-time nonlinear systems, the fast subsystem can converge to equilibrium [13, 12, 9] or limit cycles [1, 10, 25, 6]. However, for discrete-time nonlinear systems, the existing results have only focused on the case when the fast subsystem has a unique equilibrium point [20, 4, 3, 19, 28, 27]. To the best of authors' knowledge, the existence of limit cycles in the fast subsystem has not been considered, though many engineering systems exhibit limit cycles. The study of limit cycles has "long history" for discrete-time systems with applications from population models and Lienard systems [24, 8], digital filters [14, 15], digital phase-locked loops [18], discrete-time systems with power rule [26, 16] and so on. Thus it is important to extend the existing singular perturbation results for nonlinear discrete-time dynamics, which have the fast subsystem admitting limit cycles.

This work extends the singular perturbation techniques for a class of discrete-time nonlinear systems whose the fast subsystem admits a limit cycles. The main result (Theorem 1) presents the closeness of solutions between the original system, the boundary layer system, and the reduced system on a finite time interval. With the help of Theorem 1, Corollary 1 concludes the stability property of the singularly perturbed system from the stability properties of the reduced system and the boundary layer system.

This paper is organized as follows. Section 2 presents the needed preliminaries and the problem formulation. Section 3 provides the main results. Two simulation examples validate the obtained results in Section 4.

II. Preliminaries and Problem Formulation

A. Preliminaries

Let $\mathcal{Z}_+$ be the set containing all non-negative integers. The notation $\mathcal{R}$ represents the set of all real numbers. For any vector $x \in \mathcal{R}^n$, $|x|$ represents its Euclidean norm, which is defined as $|x| \triangleq \sqrt{x^\intercal x}$, where $(\cdot)^\intercal$ represents the transpose. $|x|_\mathcal{A}$ represents the distance from a point x to a closed set $\mathcal{A}$ that $|x|_\mathcal{A} = \inf_{\zeta \in A} |\zeta - x|$. $d_H(\mathcal{A}, \mathcal{B})$ is the Hausdorff distance between two subsets $\mathcal{A}$ and $\mathcal{B}$. The set B_r is defined as $B_r := \{x \in \mathcal{R}^n \,||x| \leq r\}$. The set containing all essentially bounded measurable functions is denoted as $\mathcal{L}_\infty$. For any $w(\cdot) \in \mathcal{L}_\infty$ its norm is defined $\|w\|_\infty = \sup_{k\geq 0} |w(k)|$, $k \in \mathcal{Z}_+$.

A continuous function γ: $\mathcal{R}_{\geq 0} \to \mathcal{R}_{\geq 0}$ belongs to class-$\mathcal{K}$ if it is strictly increasing and $\gamma(0) = 0$. It is of class-$\mathcal{K}_\infty$ if it belongs to class-$\mathcal{K}$ and becomes unbounded as its argument becomes unbounded. A function $\beta : \mathcal{R}_{\geq 0} \times \mathcal{R}_{\geq 0} \to \mathcal{R}_{\geq 0}$ is of class-$\mathcal{KL}$ if $\beta(\cdot, t)$ belongs to class-$\mathcal{K}$ for each $t \geq 0$ and $\beta(s, \cdot)$ decreases to zero as its argument $\to \infty$ for each s [12, Chapter 3].

The following class of discrete-time nonlinear systems are considered:

$$x(k+1) = f(x(k)), \tag{1}$$

where state $x(k) \in \mathcal{R}^n$, and for all $k \in \mathcal{Z}_+$. Next the definition of limit sets is provided, followed by the stability properties defined when (1) exhibits limit cycles.

Definition 1: (Limit set) [11] A point $P \in D$ is a limit point of the trajectory $s_k(x)$ of (1) if there exists a monotonic integer sequence $\{k_n\}_{n=0}^\infty$, with $k_n \to \infty$ as $n \to \infty$, such

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that $s_{k_n} \to P$ as $n \to \infty$. The set of all limit points of $s_k(x)$, $k \in \mathcal{Z}^+$ is the limit set $w(x)$ of $s_k(x)$ for system (1).

Definition 2: (Periodic orbit) [17] An orbit $O(x_0) = \{x_0, x_1, x_2, ...\}$ of (1) is said to be periodic with a period of $p \geq 2$ if the following holds

$$x_p = x_0 \ and \ x_i \neq x_0, \ for \ 1 \leq i \leq p-1.$$

Definition 3: (Limit cycle) [17] An orbit $O(x_0)$ of (1) is said to be asymptotically periodic if its limit set is a periodic orbit.

Remark 1: The limit set defined in Definition 3 is an asymptotically stable limit cycle. If a compact set $\mathcal{H}$, which contains the limit cycle, is defined, such a compact set satisfies the following inequality:

$$|x(k)|_{\mathcal{H}} \leq \beta(|x(0)|_{\mathcal{H}}, k), \tag{2}$$

for some $\beta \in \mathcal{KL}$. ◦

B. Problem formulation

This paper focuses on the following discrete-time nonlinear system with two time scales for any $k \in \mathcal{Z}_+$:

$$\begin{aligned} x(k+1) =& x(k) + \epsilon f(x(k), z(k)), \quad x(0) = x_0 \in \mathcal{R}^n, \\ z(k+1) =& g_1(x(k), z(k)), \quad z(0) = z_0 \in \mathcal{R}^m, \end{aligned} \tag{3}$$

where the positive small parameter ϵ satisfies $\epsilon \in (0, \epsilon_0)$ for some positive constant ϵ_0.

The system (3) has both fast dynamics (z-subsystem) and slow dynamics (x-subsystem) when ϵ is sufficiently small.

For the convenience of further analysis, we denote that $g(x(k), z(k)) = g_1(x(k), z(k)) - z(k)$. Now system (3) can be re-written as:

$$\begin{aligned} x(k+1) =& x(k) + \epsilon f(x(k), z(k)), \quad x(0) = x_0, \\ z(k+1) =& z(k) + g(x(k), z(k)), \quad z(0) = z_0. \end{aligned} \tag{4}$$

The following assumption is widely used for the nonlinear mappings f and g in (4).

Assumption 1: The nonlinear mappings $f : \mathcal{R}^n \times \mathcal{R}^m \to \mathcal{R}^n$ and $g : \mathcal{R}^n \times \mathcal{R}^m \to \mathcal{R}^m$ are locally Lipschitz with respect to x, z.

Remark 2: For continuous-time nonlinear systems, this assumption is always needed to ensure the existence and uniqueness of solutions [12, Chapter 3]. For discrete-time nonlinear systems (4), the condition for existence of solutions is much weaker than its continuous-time counterpart. In this work, we use this assumption to show the closeness of solutions. Our future work will relax this assumption. ◦

Remark 3: For continuous-time nonlinear systems [22, 10], the condition of forward completeness [23, Definition 3] is required to prevent the possible finite escape phenomenon from happening. Such a condition is not needed for discrete-time dynamics. ◦

Similar to the standard singular perturbation technique [12, 2], we first "freeze" x, i.e., $\epsilon = 0$, in (4), $x(k+1) = x(k) = x$. That is, the slow state $x(k)$ becomes a constant vector $x \in \mathcal{R}^n$. This leads to the following boundary layer system:

$$z_b(k+1) = z_b(k) + g(x, z_b(k)), \quad z_b(0) = z_0. \tag{5}$$

It is assumed that the boundary layer system (5) admits the limit cycle parameterized by the constant x with some nice boundedness property as shown in the following assumption.

Assumption 2: For any $R > 0$, the set $\mathcal{H}_x$ that contains the parameterized limit cycle is bounded for any $x \in B_R$. For a given positive constant r_b, we define a compact set $\mathcal{M}^x_{r_b}$ as $\mathcal{M}^x_{r_b} := \left\{z \in \mathcal{R}^m \left| |z|_{\mathcal{H}_x} \leq r_b \right.\right\}$. It is assumed that the boundary layer system has local asymptotic stability.

Assumption 3: Let (r, r_b) be any given positive pair. The trajectories of (5) represented as $z_b(k, x, z_0)$ asymptotically converge to the parameterized limit cycle $\mathcal{H}_x$ for all $x \in B_r$. More precisely, there exists a $\beta_b \in \mathcal{KL}$ such that for any $k \in \mathcal{Z}_+$, the following holds:

$$|z_b(k, x, z_0)|_{\mathcal{H}_x} \leq \beta_b(|z_0|_{\mathcal{H}_x}, k), \tag{6}$$

for all $z_0 \in \mathcal{M}^x_{r_b}$. Moreover, for any $r \geq 0$ and $x_1, x_2 \in B_r$, there exists $L_H > 0$ such that the distance measure $d_{\mathcal{H}}(\mathcal{H}_{x_1}, \mathcal{H}_{x_2})$ satisfies the following condition:

$$d_{\mathcal{H}}(\mathcal{H}_{x_1}, \mathcal{H}_{x_2}) \leq L_H \left| x_1 - x_2 \right|. \tag{7}$$

Remark 4: The condition (6) states that the fast dynamics asymptotically converge to a limit cycle. Such an assumption has been used for singularly perturbed systems in continuous-time when the fast dynamics admits limit cycles, for example, [10, 1, 22]. Similarly, similar to its continuous-time counterpart, condition (7) requires that the distance between two limit cycles ($x = x_1$ and $x = x_2$) needs to have some Lipschitz continuity as discussed in [6]. ◦

Similar to continuous-time cases [22, 10, 6], when fast discrete-time dynamics have limit cycles, the reduced discrete-time dynamics might have fast time-varying components. Thus the averaging techniques can be used to simplify the stability analysis of the reduced system. Assumption 4 admits the existence of a well-defined average f_{av} with respect to f.

Assumption 4: For the trajectory of the system (5) $z_b(k, x, z_0)$ starting from any $x \in \mathcal{R}^n$, $z_0 \in \mathcal{R}^m$, there is a locally Lipschitz continuous function $f_{av}(x) : \mathcal{R}^n \to \mathcal{R}^n$ such that there exist $\beta_{av} \in \mathcal{KL}$ and $N^* \in \mathcal{Z}_+$ such that for all $N \geq N^*$, the following inequality holds:

$$\begin{aligned} &\left| f_{av}(x) - \frac{1}{N} \sum_{k=0}^{N-1} f(x, z_b(k, x, z_0)) \right| \\ \leq \quad & \beta_{av}(\max\{|x|, \|z_b\|_\infty, 1\}, N). \end{aligned} \tag{8}$$

With this assumption, the following reduced averaged system is obtained:

$$x_r(k+1) = x_r(k) + \epsilon f_{av}(x_r(k)), \quad x_r(0) = x_0. \tag{9}$$

Sometimes, we also call (9) as the reduced system when no confusion will arise.

Remark 5: In general, when a family of limit cycles exist, we should define the reduced system (9) using difference inclusion $x_r(k+1) \in x_r(k) + \epsilon F_{av}(x_r(k))$ where

$$F_{av}(x) = \bigcup_{z_0 \in \mathcal{M}^x_{r_b}} \{f_{av}(x)\}. \tag{10}$$

For the simplicity of presentation, this paper only considers the case when only one limit cycle exists and the set value map $F_{av}(x)$ is singleton. The similar analysis can be extended to multiple limit cycles. $\circ$

Remark 6: A speical case of our result is that the boundary layer system (5) converges to a unique equilibrium as shown in [7, 20, 3].

III. Main Results

This section discusses the closeness of solutions between the discrete-time nonlinear system (4) (solutions denoted as $(x(k), z(k))$), its reduced system (9) (solutions denoted as $x_r(k)$), and its boundary layer system (5) (solutions denoted as $z_b(k)$) on a finite interval $k \in [0, [\frac{N}{\epsilon}]]$[1], for any $N \in \mathcal{Z}_+$.

We divide the interval $[0, [\frac{N}{\epsilon}]]$ into sub-intervals of the form $[k_l, k_{l+1}]$ where l is the element of the index set $I_\epsilon = \{0, \ldots, [\frac{N}{\epsilon h_\epsilon}]\}$. Except for the last sub-interval, each sub-interval has the same length h_ϵ. The nonlinear mapping h_ϵ is a function of ϵ satisfying $\lim_{\epsilon\to 0} h_\epsilon = \infty$ and $\lim_{\epsilon\to 0} \epsilon h_\epsilon = 0$.

With an initial value $(x_0, z_0) \in \mathcal{R}^n \times \mathcal{R}^m$, the solutions of (4) for the interval $k \in [k_l, k_{l+1})$ are:

$$x(k) = x(k_l) + \epsilon \sum_{i=k_l}^{k-1} f(x(i), z(i)), \tag{11}$$

$$z(k) = z(k_l) + \sum_{i=k_l}^{k-1} g(x(i), z(i)), \tag{12}$$

while the solutions of (9) take the following form:

$$x_r(k) = x_r(k_l) + \epsilon \sum_{i=k_l}^{k-1} f_{av}(x_r(i)). \tag{13}$$

Let $\xi(0) = x_0$, $\eta(0) = z_0$, and $\omega(0) = x_0$, three auxiliary series $\xi(k)$, $\eta(k)$ and $\omega(k)$ are used in the analysis:

$$\xi(k) = \xi(k_l) + \epsilon \sum_{i=k_l}^{k-1} f(\xi(k_l), \eta(i)), \tag{14}$$

$$\eta(k) = \eta(k_l) + \sum_{i=k_l}^{k-1} g(\xi(k_l), \eta(i)), \eta(k_l) = z(k_l), \tag{15}$$

$$\omega(k) = \omega(k_l) + \epsilon \sum_{i=k_l}^{k-1} f_{av}(\xi(k_l)), \omega(k_l) = x_r(k_l). \tag{16}$$

It is noted that for each $k \in [k_l, k_{l+1})$, $\eta(k)$ is the solution of the boundary layer system (5) with a fixed $x = \xi(k_l)$ and an initial value of $\eta(k_l) = z(k_l)$.

The main result is presented as follows.

Theorem 1: Assume that Assumptions $1-4$ hold for the system (4). For any positive real triple (r, r_b, δ) and any positive integer N, there is a positive ϵ^* such that for any $\epsilon \in (0, \epsilon^*)$, any integer $k \in \left[0, \left[\frac{N}{\epsilon}\right]\right]$, the solutions of (4), (5), and (9) satisfy

$$|x(k) - x_r(k)| \le \delta, \tag{17}$$

$$|z(k)|_{\mathcal{H}_{x(k)}} \le \beta_b(|z_0|_{\mathcal{H}_{x_0}}, k) + \delta, \tag{18}$$

where $x_0 \in B_r$, $z_0 \in \mathcal{M}^{x_0}_{r_b}$.

[1] $[\frac{N}{\epsilon}]$ denotes the largest integer i such that $i \le \frac{N}{\epsilon}$

This proof follows a similar steps used in [22, 6] for continuous-time systems.

Proof: For the discrete-time system (4), its boundary layer system (5), and its reduced system (9), the existence and uniqueness of the solutions for any given finite time interval are guaranteed by the local Lipschitz continuity properties of f, g, and f_{av} respectively when the initial conditions x_0, z_0 are bounded. Consequently, there exist $R > r$, $R_b > r_b$ such that $x(k), \xi(k), \omega(k), x_r(k) \in B_R$, $z(k), \eta(k) \in \bigcup_{x\in B_R} \mathcal{M}^x_{R_b} \in \mathcal{M}$ for any $k \in [0, [\frac{N}{\epsilon}]]$, where $\mathcal{M}$ is a compact set.

From Assumption 1, L is denoted as the largest Lipschitz constant of f and g on $B_R \times \mathcal{M}$ while L_{av} is the Lipschitz constant of f_{av} on B_R. Also, there exists $P > 0$ that $\max_{x\in B_R, y\in\mathcal{M}} \{|f(x,y)|, |g(x,y)|, |f_{av}(x)|\} \le P$. From Assumption 2, the constant L_H is the Lipschitz constant from the inequality (7).

<u>The first part will prove (17)</u>

For convenience, the following notations are used

$$\alpha(k) = |z(k) - \eta(k)|, \bar{\alpha} = \max_{0\le k\le[\frac{N}{\epsilon}]} \alpha(k), \tag{19}$$

$$\phi(k) = |x(k) - \xi(k)|, \bar{\phi} = \max_{0\le k\le[\frac{N}{\epsilon}]} \phi(k). \tag{20}$$

The difference between two solutions is considered

$$|x(k) - x_r(k)| \le |x(k) - \xi(k)| + |\xi(k) - \omega(k)| + |\omega(k) - x_r(k)|. \tag{21}$$

We can bounded three parts in (21) separately.

Step 1 shows the closeness of solutions between z and η.

For $k \in [k_l, k_{l+1})$, using $\eta(k_l) = z(k_l)$, by using (12), (15) and (19), it follows that

$$\begin{aligned}\alpha(k) &= \sum_{i=k_l}^{k-1} |g(x(i), z(i)) - g(\xi(k_l), \eta(i))| \\ &\le \sum_{i=k_l}^{k-1} L\left(|x(i) - \xi(k_l)| + |z(i) - \eta(i)|\right) \\ &\le \sum_{i=k_l}^{k-1} L(|x(k_l) - \xi(k_l) + x(i) - x(k_l)| \\ &\quad + |z(i) - \eta(i)|) \\ &\le \sum_{i=k_l}^{k-1} L(|x(k_l) - \xi(k_l)| + |x(i) - x(k_l)|) \qquad (22) \\ &\quad + \sum_{i=k_l}^{k-1} L\alpha(i), \qquad (23)\end{aligned}$$

where (22) can be further bounded by

$$\begin{aligned}&\sum_{i=k_l}^{k-1} L(|x(k_l) - \xi(k_l)| + |x(i) - x(k_l)|) \\ &\le L\left(\sum_{i=k_l}^{k-1} \phi(k_l) + \sum_{i=k_l}^{k-1}\sum_{s=k_l}^{i-1} \epsilon f(x(s), z(s))\right) \\ &\le L h_\epsilon\left(\phi(k_l) + h_\epsilon \epsilon P\right), \qquad (24)\end{aligned}$$

from the continuity of f.

The second term of $\alpha(k)$, i.e. (23), by (24) and discrete-time Gronwall lemma [5], for $k \in [k_l, k_{l+1})$, we have

$$\alpha(k) \leq L h_\epsilon \left(\phi(k_l) + h_\epsilon \epsilon P\right) e^{L h_\epsilon} \leq \bar{\alpha}, \tag{25}$$

where

$$\bar{\alpha} = L h_\epsilon \left(\bar{\phi} + h_\epsilon \epsilon P\right) e^{L h_\epsilon}. \tag{26}$$

Step 2 shows the closeness of solutions between x and ξ. For $k \in [k_l, k_{l+1})$, using (11) and (14), we have

$$\begin{aligned}\phi(k) \leq& \phi(k_l) + \epsilon \sum_{i=k_l}^{k-1} |f(x(i), z(i)) - f(\xi(k_l), \eta(i))| \\ \leq& \phi(k_l) + \epsilon \sum_{i=k_l}^{k-1} L(|x(i) - \xi(k_l)| + |z(i) - \eta(i)|) \\ \leq& \phi(k_l) + \epsilon L \sum_{i=k_l}^{k-1} (|x(k_l) - \xi(k_l)| \\ &+ |x(i) - x(k_l)| + \alpha(i)),\end{aligned} \tag{27}$$

which can be further bounded by using (24) and (25):

$$\begin{aligned}\phi(k) \leq& \phi(k_l) + \epsilon L h_\epsilon \left(\phi(k_l) + h_\epsilon \epsilon P\right. \\ &+ L h_\epsilon (\phi(k_l) + h_\epsilon \epsilon P) e^{L h_\epsilon} \\ =& (1 + \epsilon L h_\epsilon (1 + L h_\epsilon e^{L h_\epsilon}))\phi(k_l) \\ &+ \epsilon L h_\epsilon (h_\epsilon \epsilon P + L h_\epsilon h_\epsilon \epsilon P e^{L h_\epsilon}) \\ =& (1 + \epsilon h_\epsilon a)\phi(k_l) + \epsilon h_\epsilon \epsilon b,\end{aligned} \tag{28}$$

where

$$\begin{aligned}a =& L(1 + L h_\epsilon e^{L h_\epsilon}), \\ b =& L(h_\epsilon P + L h_\epsilon h_\epsilon P e^{L h_\epsilon}).\end{aligned} \tag{29}$$

As $\phi(0) = |x(0) - \xi(0)| = 0$, by using Induction, for $k \in [k_l, k_{l+1})$, it yields

$$\begin{aligned}\phi(k) \leq& (1 + \epsilon h_\epsilon a)^l \phi(0) + \frac{(1 + (1 + \epsilon h_\epsilon a)^{l-1})\epsilon h_\epsilon \cdot \epsilon l \cdot b}{2} \\ \leq& \frac{(1 + (1 + \epsilon h_\epsilon a)^{l-1})\epsilon h_\epsilon \cdot \epsilon l \cdot b}{2}.\end{aligned} \tag{30}$$

Now we define mapping h_ϵ such that $\lim_{\epsilon \to 0} \epsilon h_\epsilon a = 0$. Since $l \in \{0, ..., [\frac{N}{\epsilon h_\epsilon}]\}$, we have

$$\begin{aligned}\bar{\phi} \leq& \frac{(1 + (1 + \epsilon h_\epsilon a)^{[\frac{N}{\epsilon h_\epsilon}] - 1})\epsilon h_\epsilon [\frac{N}{\epsilon h_\epsilon}] \epsilon b}{2} \\ \leq& \frac{(1 + (1 + \epsilon h_\epsilon a)^{\frac{1}{\epsilon h_\epsilon a} N a})\epsilon N b}{2} \\ =& \epsilon \frac{(1 + e^{N a}) N b}{2}.\end{aligned} \tag{31}$$

By (31), for a given δ, there is a ϵ_1^* that for any $\epsilon \in (0, \epsilon_1^*)$, $\bar{\phi} \leq \frac{\delta}{3}$.

Also, with the consideration of (30) in (26), with small small ϵ, $\bar{\alpha}$ will approach to 0. This property will be used for the proof of (18).

Step 3 shows the closeness of solutions between ξ and ω. For $k \in [k_l, k_{l+1})$, using (14) and (16), we have

$$\begin{aligned}|\xi(k) - \omega(k)| \leq& |\xi(k_l) - \omega(k_l)| \\ &+ \epsilon \sum_{i=k_l}^{k-1} |f(\xi(k_l), \eta(i)) - f_{av}(\xi(k_l))|.\end{aligned} \tag{32}$$

Since $\eta(k)$ is the solution of (5) with fixed $x = \xi(k_l)$ and initial $\eta(k_l) = z(k_l)$, we can direct use (8) in Assumption 4 for (32).

Let us check $k = k_l$. By (6) in Assumption 3 and (8) in Assumption 4, by choosing $h_\epsilon > N^*$, it results in

$$\begin{aligned}&|\xi(k_{l+1}) - \omega(k_{l+1})| \leq |\xi(k_l) - \omega(k_l)| \\ &+ \epsilon h_\epsilon \beta_{av}(\max\{|\xi(k_l)|, \|\eta\|_\infty\}, h_\epsilon).\end{aligned} \tag{33}$$

Since $\xi(0) = \omega(0) = x_0$, by Induction, for $l \in I_\epsilon$, the following inequality holds:

$$\begin{aligned}|\xi(k_l) - \omega(k_l)| \leq& [\frac{N}{\epsilon}] \epsilon \beta_{av}(\max\{R, d\}, h_\epsilon) \\ \leq& N \beta_{av}(\max\{R, d\}, h_\epsilon),\end{aligned} \tag{34}$$

where $d = \max_{\eta \in \mathcal{M}} |\eta|$.

Next will check $k \in (k_l, k_{l+1})$. Using (32) and the boundedness of f and f_{av} yields

$$|\xi(k) - \omega(k)| \leq N \beta_{av}(\max\{R, d\}, h_\epsilon) + 2\epsilon h_\epsilon P. \tag{35}$$

And (35) holds for $k \in [0, [\frac{N}{\epsilon}]]$.

For a given δ, we choose a sufficiently large h_ϵ, and there exist an ϵ_2^* such that for $(0, \epsilon_2^*)$, $|\xi(k) - \omega(k)| \leq \frac{\delta}{3}$.

Step 4 shows the closeness of solution between ω and x_r.

Using (16), (13), $\omega(0) = x_r(0) = x_0$, and $\omega(k_l) = x_r(k_l)$, for $k \in [k_l, k_{l+1})$, it can be shown that

$$\begin{aligned}|\omega(k) - x_r(k)| \leq& \epsilon \sum_{i=k_l}^{k-1} |f_{av}(\xi(k_l)) - f_{av}(x_r(i))| \\ \leq& \epsilon L_{av} \sum_{i=k_l}^{k-1} |\xi(k_l) - x_r(i)| \\ \leq& \epsilon L_{av} \sum_{i=k_l}^{k-1} (|\xi(k_l) - \omega(k_l)| \\ &+ |\omega(k_l) - \omega(i)| + |\omega(i) - x_r(i)|).\end{aligned} \tag{36}$$

With the help of (16), (34), it follows that

$$\begin{aligned}|\omega(k) - x_r(k)| \leq& \epsilon L_{av} \sum_{i=k_l}^{k-1} \big(N \beta_{av}(\max\{R, d\}, h_\epsilon) \\ &+ \epsilon \sum_{s=k_l}^{i-1} |f_{av}(\xi(k_l))|\big) + \epsilon L_{av} \sum_{i=k_l}^{k-1} |\omega(i) - x_r(i)| \\ \leq& \epsilon L_{av} h_\epsilon (N \beta_{av}(\max\{R, d\}, h_\epsilon) + \epsilon h_\epsilon P) \\ &+ \epsilon L_{av} \sum_{i=k_l}^{k-1} |\omega(i) - x_r(i)|.\end{aligned} \tag{37}$$

By using discrete-time Gronwall Lemma [5], we have

$$|\omega(k) - x_r(k)| \leq \epsilon L_{av} h_\epsilon \big(N\beta_{av}(\max\{R, d\}, h_\epsilon) + \epsilon h_\epsilon P\big) e^{\epsilon h_\epsilon P}. \tag{38}$$

And (38) holds for $k \in [0, [\frac{N}{\epsilon}]]$. The analysis for (38) is similar to (35), leading to a ϵ_3^* for a given δ.

Step 5 concludes the result by combining (31), (35), (38) from steps $2-4$, leading to

$$\begin{aligned} |x(k) - x_r(k)| \leq& \bar{\phi} + \max|\xi(k) - \omega(k)|\,| \\ &+ \max|\omega(k) - x_r(k)| \\ \leq& \delta. \end{aligned} \tag{39}$$

The second part will prove (18)

It is noted that for each $k \in [k_l, k_{l+1}]$, the notion of $\eta(k)$ are the solutions of boundary layer system (5) $z_b(k, \xi(k_l), \eta(k_l))$ with a fixed $x = \xi(k_l)$ and the initial value of $z_b[0] = \eta(k_l)$. Using (6) in Assumption 3 and $\eta(k_l) = z(k_l)$, for $k \in [k_l, k_{l+1})$, the trajectories of $\eta[k]$ satisfy

$$|\eta(k)|_{\mathcal{H}_{\xi(k_l)}} \leq \beta_b(|z(k_l)|_{\mathcal{H}_{\xi(k_l)}}, k - k_l). \tag{40}$$

By $\alpha(k) = |z(k) - \eta(k)|$ and $\bar{\alpha} = \max_{0 \leq k \leq [\frac{N}{\epsilon}]} \alpha(k)$, this results in

$$\begin{aligned} |z(k)|_{\mathcal{H}_{\xi(k_l)}} \leq& |\eta(k)|_{\mathcal{H}_{\xi(k_l)}} + |z(k) - \eta(k)| \\ \leq& |\eta(k)|_{\mathcal{H}_{\xi(k_l)}} + \bar{\alpha}. \end{aligned} \tag{41}$$

By (7) in Assumption 3 and triangle inequality, the following inequality is obtained:

$$\begin{aligned} |z(k)|_{\mathcal{H}_{x(k)}} \leq& |z(k)|_{\mathcal{H}_{\xi(k_l)}} + d_H(\mathcal{H}_{x(k)}, \mathcal{H}_{\xi(k_l)}) \\ \leq& |z(k)|_{\mathcal{H}_{\xi(k_l)}} + L_H\,|x(k) - \xi(k_l)| \\ \leq& |z(k)|_{\mathcal{H}_{\xi(k_l)}} + L_H(|x(k) - x(k_l)| \\ &+ |x(k_l) - \xi(k_l)|) \\ \leq& |z(k)|_{\mathcal{H}_{\xi(k_l)}} + L_H(\epsilon h_\epsilon P + \bar{\phi}). \end{aligned} \tag{42}$$

By combining (40), (41), (42), we have

$$\begin{aligned} |z(k)|_{\mathcal{H}_{x(k)}} \leq& |\eta(k)|_{\mathcal{H}_{\xi(k_l)}} + \bar{\alpha} + L_H(\epsilon h_\epsilon P + \bar{\phi}) \\ \leq& \beta_b(|z(k_l)|_{\mathcal{H}_{\xi(k_l)}}, k - k_l) + \gamma \\ \leq& \beta_b(|z(k_l)|_{\mathcal{H}_{x(k_l)}} \\ &+ d_H(\mathcal{H}_{x(k_l)}, \mathcal{H}_{\xi(k_l)})), k - k_l) + \gamma \\ \leq& \beta_b(|z(k_l)|_{\mathcal{H}_{x(k_l)}} \\ &+ L_H\,|x(k_l) - \xi(k_l)|\,, k - k_l) + \gamma \\ \leq& \beta_b(|z(k_l)|_{\mathcal{H}_{x(k_l)}} + L_H\bar{\phi}, k - k_l) + \gamma \\ \leq& \beta_b(|z(k_l)|_{\mathcal{H}_{x(k_l)}}, k - k_l) + \gamma_1, \end{aligned} \tag{43}$$

where $\gamma = \bar{\alpha} + L_H(\epsilon h_\epsilon P + \bar{\phi})$, $\gamma_1 = \gamma + \kappa(L_H\bar{\phi})$, and $\kappa(\cdot)$ is a class-$\mathcal{K}$ function.

As being discussed, with large enough h_ϵ, and small enough ϵ, (γ, γ_1) can be selected to approach to 0, by using similar induction method used in [22, Theorem 1], for any $\delta > 0$, the following inequality holds:

$$|z(k)|_{\mathcal{H}_{x(k)}} \leq \beta_b(|z_0|_{\mathcal{H}_{x_0}}, k) + \delta, \tag{44}$$

which completes the proof. ■

Remark 7: Theorem 1 indicates that for a given distance $\delta > 0$, by tuning the parameter ϵ sufficiently small, the solutions of (4), and its reduced system (9), boundary layer system (5) can be made close within this distance for a given finite time interval. This result is similar to [23, 22]. Alternatively, this result shows that if ϵ is selected sufficiently small, the "error" δ will be a function of ϵ. A smaller ϵ leads to a smaller $\delta(\epsilon)$ as shown in Example 1 in simulation. ∘

Theorem 1 shows the closeness of solutions in finite time. Such closeness of solutions play a key role in analyzing the stability properties of (4). Corollary 1 summaries the stability result. The proof of this corollary is based on Induction method which can be found in [22, 10, 6], and it is omitted due to space limitation.

Corollary 1: Assume that Assumption 1 and Assumption 4 hold for the system (4). And assume that the reduced system (9) is locally or globally asymptotically stable, uniformly in small ϵ, i.e., $\exists \beta_r \in \mathcal{KL}$ such that there exists $\epsilon_r^* > 0$ such that the solutions of reduced order system satisfy

$$|x_r(k)| \leq \beta_r(|x_0|\,, \epsilon k), \tag{45}$$

for $\epsilon \in (0, \epsilon_r^*)$ and $|x_0| \leq \Delta_r$ for some $\Delta_r > 0$ (in global case, $\Delta_r = \mathcal{R}^n$). For given $r = \Delta_r$, Assumption 2 and Assumption 3 hold for some constant r_b. For any positive constant ν, r_b and $0 < \Delta < \Delta_r$, there exists a positive $\epsilon^* > 0$ such that for any $\epsilon \in (0, \epsilon^*)$, then solutions of (4) satisfy

$$|x(k)| \leq \beta_r(|x_0|\,, \epsilon k) + \nu, \tag{46}$$

$$|z(k)|_{\mathcal{H}_{x(k)}} \leq \beta_b(|z_0|_{\mathcal{H}_{x_0}}\,, k) + \nu, \tag{47}$$

for any $x_0 \in B_\Delta$ and $z_0 \in \mathcal{M}_{r_b}^{x_0}$.

For the local stability property in Corollary 1, Assumptions $1-4$ can be further relaxed, and they only need to hold in some local regions.

We can further extend our results to more general cases, for example, the input-to-state stability property of the original system (4) based on the input-to-state stability property of its limiting systems (5) and (9) using the similar ideas in continuous singular perturbation presented by [22]. This will be part of our future work.

IV. Simulation Results

In order to illustrate our theoretical findings, two simulation examples are provided. In two examples, the same boundary layer system, which has exponential stability property, is used while the reduced systems have different stability properties. First example shows the closeness of solutions (Theorem 1), in which the reduced system is stable, but not attractive. Second example shows the stability properties of overall system when the reduced system is globally exponentially stable (Corollary 1).

A. Example 1

First we show that ϵ in (4) can affect the closeness of solution in finite time. We consider a continuous-time system

$$\begin{aligned} \dot{x} &= \epsilon z_1 \\ \dot{z}_1 &= -z_1 + z_2 + \frac{x z_1}{\sqrt{z_1^2 + z_2^2}} \\ \dot{z}_2 &= -z_1 - z_2 + \frac{x z_2}{\sqrt{z_1^2 + z_2^2}}. \end{aligned} \tag{48}$$

After sampling system (48) with sampling period $T = \frac{\pi}{4}$, we write the exact discrete-time system in the format of (4). The exact discrete-time system can be obtained by switching (48) to the polar coordinates [12, Chapter 2]. In the polar coordinate, system (48) will be linear, which has the exact discretization model.

$$\begin{aligned} x(k+1) =& x(k) + \epsilon T z_1(k) \\ z_1(k+1) =& (e^T r(k) + (1 - e^T)x(k)) \cos(\theta(k) - T) \\ z_2(k+1) =& (e^T r(k) + (1 - e^T)x(k)) \sin(\theta(k) - T), \end{aligned} \tag{49}$$

where $r(k) = \sqrt{z_1^2(k) + z_2^2(k)}$, $\theta(k) = \arctan \frac{z_2(k)}{z_1(k)}$. This leads to the following boundary layer system:

$$\begin{aligned} z_{1b}(k+1) =& (e^T r(k) + (1 - e^T)x) \cos(\theta(k) - T) \\ z_{2b}(k+1) =& (e^T r(k) + (1 - e^T)x) \sin(\theta(k) - T). \end{aligned} \tag{50}$$

In polar coordinates, we can find out that for the boundary layer system (50), the exponentially stable limit cycle parametrized by x contains eight limit points $\{(x \cos(kT), x \sin(kT))\}$ for $k = 1, 2, ..., 8$. Here $\mathcal{H}_x$ in Theorem 1 contains this limit cycle. The reduced order system is stable but not attractive:

$$x(k+1) = x(k). \tag{51}$$

Now we check the property when $\epsilon = 0.1$ and 0.04. Note that when $\epsilon = 0.04$ is selected, a large time interval is needed since the speed of x is parametrized by ϵ. As Remark 7 indicates, a smaller ϵ will lead to a smaller distance between solutions of original system (4), its reduced system (9) and its boundary layer system (5). Fig. 1 shows the trajectories of original system and its reduced system with $\epsilon = 0.1$ and $\epsilon = 0.04$ respectively. On the other hand, Fig. 2 shows the closeness of solutions between the fast system and its limit cycles by using the distance measure $|z|_{\mathcal{H}_x}$. The simulation results are consistent with the result presented in Theorem 1.

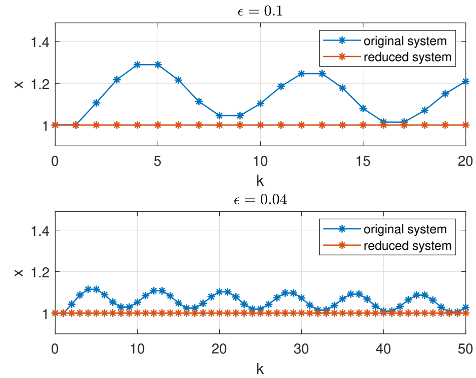

Fig. 1. Solutions of the slow variable $x(k)$ and its reduced order variable $x_r(k)$ for different choices of ϵ

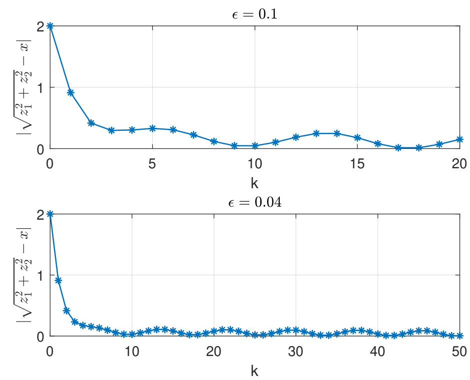

Fig. 2. Distance from fast variable $z(k)$ to its limit cycle for the difference choices of ϵ

B. Example 2

Next we will show how ϵ in (4) can affect the stability properties of the original system when the slow dynamics have nice stability properties.

The discrete time system has the form:

$$\begin{aligned} x(k+1) =& x(k) + \epsilon T(-x(k) + z(k)) \\ z_1(k+1) =& (e^T r(k) + (1 - e^T)x(k)) \cos(\theta(k) - T) \\ z_2(k+1) =& (e^T r(k) + (1 - e^T)x(k)) \sin(\theta(k) - T). \end{aligned} \tag{52}$$

While two examples share the same boundary-layer system, the following reduced system of (52) is globally exponentially stable with small enough ϵ:

$$x(k+1) = x(k) - \epsilon T x(k). \tag{53}$$

Fig.3 shows both trajectories of the slow variable $x(k)$ in (52) and (53) are convergent with different choices of ϵ, which is consistent with the result in Corollary 1.

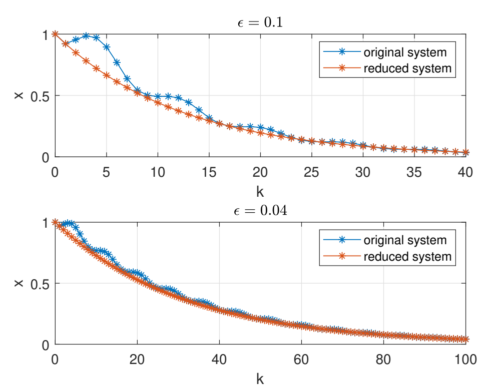

Fig. 3. Solutions of the slow variable $x(k)$ and its reduced order variable $x_r(k)$ for different choices of ϵ

V. Conclusions

In this paper, we studied the property of a class of discrete-time nonlinear systems with two time-scales when the fast dynamics admit limit cycles. The main result shows that the solutions of the original system, its boundary layer system, and its reduced order system can be made arbitrarily close on a finite interval by tuning time-scale separation parameter ϵ sufficiently small with appropriate assumptions. This result also lead to conclusion of stability properties of the original system from appropriate stability properties of its boundary layer system and its reduced order system, Our future work will consider input-to-state stability property of such system.

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