Higher Spin; Symmetry properties of anti-symmetrized gamma matrices

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Let γ_{n_i}, where i runs from 1 to d, be the gamma matrices, obeying the Clifford algebra $\{\gamma_n, \gamma_m\} = 2\eta_{mn}\mathbb{1}$. For the matrix components of the fully anti-symmetrized product of p gamma matrices we write

$$\gamma_{[n_1,\ldots,n_p]\hat{\alpha}\hat{\beta}},$$

we are now asked to find the symmetry properties of these matrices, i.e. find $\eta(p,d)$, such that the following formula holds

$$\gamma_{[n_1,\ldots,n_p]\hat{\alpha}\hat{\beta}} = \eta^(p,d)\gamma_{[n_1,\ldots,n_p]\hat{\beta}\hat{\alpha}}.$$

Where $\eta(0,d)$ is reserved for the symmetry of the charge conjugation matrix in d dimensions.

The case $p > 0$.

Let us start by noting the following crucial identities given in the lecture notes:

$$C_{\hat{\beta}\hat{\gamma}}\gamma_{n\hat{\alpha}}{}^{\hat{\gamma}} = -\eta C_{\hat{\alpha}\hat{\gamma}}\gamma_{n\hat{\beta}}{}^{\hat{\gamma}},$$
$$A_{\hat{\alpha}\hat{\beta}} := C_{\hat{\beta}\hat{\gamma}}A_{\hat{\alpha}}{}^{\gamma},$$

here C is the so called Charge conjugation matrix and A is any element of the Clifford Algebra. Furthermore $\eta := \eta(0,d)$, that is $C_{\hat{\alpha}\hat{\beta}} = \eta C_{\hat{\beta}\hat{\alpha}}$. Now let us do an explicit example,

$$\begin{aligned}\gamma_{[nm]\hat{\alpha}\hat{\beta}} &= \gamma_{n\hat{\alpha}}{}^{\hat{\epsilon}}\gamma_{m\hat{\epsilon}}{}^{\hat{\delta}}C_{\hat{\beta}\hat{\delta}} - (n \leftrightarrow m) = -\gamma_{n\hat{\alpha}}{}^{\hat{\epsilon}}\gamma_{m\hat{\beta}}{}^{\hat{\delta}}C_{\hat{\delta}\hat{\epsilon}} - (n \leftrightarrow m)\\ &= \eta\gamma_{n\hat{\delta}}{}^{\hat{\epsilon}}\gamma_{m\hat{\beta}}{}^{\hat{\delta}}C_{\hat{\alpha}\hat{\epsilon}} - (n \leftrightarrow m) = \eta(\gamma_m\gamma_n)_{\hat{\beta}\hat{\alpha}} - (n \leftrightarrow m)\\ &= -\eta\gamma_{[nm]\hat{\beta}\hat{\alpha}}.\end{aligned}$$

By using tricks as shown above it can be shown that the following formula holds,

$$(\gamma_{n_1}\gamma_{n_2}...\gamma_{n_{p-1}}\gamma_{n_p})_{\hat{\alpha}\hat{\beta}} = \eta(-1)^p(\gamma_{n_p}\gamma_{n_{p-1}}...\gamma_{n_2}\gamma_{n_1})_{\hat{\beta}\hat{\alpha}}. \tag{1}$$

Proving this via index-algebraic manipulations as shown above is quite tedious. Alternatively one can show this using Penrose graphical notation (or some variation thereof?). This necessitates the translations of the given properties of the charge conjugation into graphical notation, once this is done the rest follows quite naturally. Now given that we are interested in fully anti-symmetrised products we naturally fulfil the requirements for Eq. (1) to hold. Inspecting Eq. (1) more closely we still see that we need to permute a product of p different gamma matrices to flip the order. To do so we need to do $\sum_{n=1}^{p-1}(p-n) = p(p-1)/2$ permutations. To bring the gamma matrix on the far left to the right we need to do $p-1$ permutations, the next one needs $p-2$ permutations, etc. we are done when we have done $p-1$ permutations, this leads to the previous expression. Symbolically this looks as follows, (keeping in mind that each permutation brings a factor of (-1))

$$\begin{aligned}&\gamma_1\gamma_2...\gamma_p = (-1)^{p-1}\gamma_2...\gamma_p\gamma_1 = (-1)^{p-1+p-2}\gamma_3...\gamma_p\gamma_2\gamma_1 \\ &= ... = (-1)^{\sum_{n=1}^{p-1}(p-n)}\gamma_p...\gamma_2\gamma_1,\end{aligned} \tag{2}$$

note that again we have assumed that $n_i \neq n_j$ for all $i \neq j$. Combining this with Eq. (1) we obtain

$$\begin{aligned}(\gamma_{n_1}\gamma_{n_2}...\gamma_{n_p})_{\hat{\alpha}\hat{\beta}} &= \eta(-1)^p(-1)^{\frac{1}{2}(p^2-p)}(\gamma_{n_1}\gamma_{n_2}...\gamma_{n_p})_{\hat{\beta}\hat{\alpha}} \\ &= \eta(-1)^{\frac{1}{2}(p^2+p)}(\gamma_{n_1}\gamma_{n_2}...\gamma_{n_p})_{\hat{\beta}\hat{\alpha}},\end{aligned}$$

this can be immediately applied to the anti-symmetrized products we are interested in i.e.

$$\gamma_{[n_1...n_p]\hat{\alpha}\hat{\beta}} = \eta(0,d)(-1)^{\frac{1}{2}(p^2+p)}\gamma_{[n_1...n_p]\hat{\beta}\hat{\alpha}}.$$

Note that if $n_i = n_j$ for any pair $i \neq j$ we have that both sides of the equation are zero, so we may assume $n_i \neq n_j$ for all $i \neq j$, thus fulfilling the sufficient conditions to use Eqs. (1,2).

The even dimensional case

In the following we will restrict ourselves to the case that the dimension is even, since the case of odd dimension suffers from some peculiarities with the maximal element Γ, and in any case it can be retrieved as the direct sum of even dimensional cases.

The table below shows the number of matrices that are even/symmetric and odd/anti-symmetric as a function of dimension and as a function of the sign of $\eta(0, d)$.

d	d=2	d=4	d=6	d=8	d=10	d=12	d=14
# even, η=+1	1	6	36	136	496	2016	8256
# odd, η=+1	3	10	28	120	528	2080	8128
# even, η=-1	3	10	28	120	528	2080	8128
# odd, η=-1	1	6	36	136	496	2016	8256

The table above was constructed in the following manner. We know that that $\eta(p, d) = (\eta(0, d))(-1)^{\frac{1}{2}(p^2+p)}$, now suppose that we know the dimension d, the number of matrices of order p then is equal to d choose p, or $d!/((d-p)!p!)$, this means that at given even d and given $\eta(0, d)$, we can find the total number of symmetric matrices as follows

$$\#\mathrm{even}(d) = \frac{\eta(0, d) + 1}{2} + \sum_{p=1}^{d} \left(\frac{\eta(0, d)(-1)^{\frac{1}{2}(p^2+p)} + 1}{2} \right) \binom{d}{p},$$

and the total number of anti-symmetric matrices is given by

$$\#\mathrm{odd}(d) = -\frac{\eta(0, d) - 1}{2} - \sum_{p=1}^{d} \left(\frac{\eta(0, d)(-1)^{\frac{1}{2}(p^2+p)} - 1}{2} \right) \binom{d}{p},$$

as mentioned above, these formulae are valid for even d, however for odd d we already know the symmetry property of $\eta(0, d)$.

The table above suggest the following pattern: In $d = 2$ we must choose $\eta = -1$, since the number of symmetric(/even) matrices must be higher than the number of anti-symmetric(/odd) matrices. In $d = 4$ we have $\eta = -1$, in $d = 6$ we have $\eta = +1$ and in $d = 8$ we have $\eta = +1$. Dimensions higher than these, we need not consider because of the period 8 Bott periodicity. It is not difficult two write down a formula that satisfies this pattern, for example we could have $\eta(0, d) = (-1)^{(d^2+2d)/8}$, however it is not clear if this is useful because we do not know the behaviour of η for odd dimensions, (it is however clear that this example would not work in odd dimensions since there it will not be a root of 1).

Check for $d = 1, 2$

In the case that $d = 1$ we have $\eta(p, 1) = \eta(0, 1)$, so we see that $\eta(0, 1)$ should be $+1$ since the space of anti-symmetric one by one matrices is trivial.

Now let us consider the case $d = 2$, here the two matrices generating the Clifford algebra are σ_1 and σ_2. We give the explicit forms of the relevant matrices

$$\sigma_{1\hat{\alpha}}{}^{\hat{\beta}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_{2\hat{\alpha}}{}^{\hat{\beta}} = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \qquad C_{\hat{\alpha}\hat{\beta}\pm} = \begin{pmatrix} C_{11} & C_{12} \\ \pm C_{12} & C_{22} \end{pmatrix},$$

we keep track of the cases that C is either anti-symmetric or symmetric. Let us construct the matrix form of $\sigma_{1\hat{\alpha}\hat{\beta}}$

$$\sigma_{1\hat{\alpha}\hat{\beta}} = C_{\hat{\beta}\hat{1}}\sigma_{1\hat{\alpha}}{}^{\hat{1}} + C_{\hat{\beta}\hat{2}}\sigma_{1\hat{\alpha}}{}^{\hat{2}} = \begin{pmatrix} C_{\hat{1}\hat{2}} & C_{\hat{2}\hat{2}} \\ C_{\hat{1}\hat{1}} & C_{\hat{2}\hat{1}} \end{pmatrix},$$

doing the same thing for $\sigma_{2\hat{\alpha}\hat{\beta}}$ we get

$$\sigma_{2\hat{\alpha}\hat{\beta}} = C_{\hat{\beta}\hat{1}}\sigma_{2\hat{\alpha}}{}^{\hat{1}} + C_{\hat{\beta}\hat{2}}\sigma_{2\hat{\alpha}}{}^{\hat{2}} = \begin{pmatrix} iC_{\hat{1}\hat{2}} & iC_{\hat{2}\hat{2}} \\ -iC_{\hat{1}\hat{1}} & -iC_{\hat{2}\hat{1}} \end{pmatrix}.$$

Note that from the equations above we cannot derive any relationship between $C_{\hat{1}\hat{2}}$ and $C_{\hat{2}\hat{1}}$, so the symmetry of C is still an open question. We know the commutation relations of the Pauli spin matrices, they tell us that $\sigma_{[12]\hat{\alpha}}{}^{\hat{\beta}}$ is proportional to $\sigma_{3\hat{\alpha}}{}^{\hat{\beta}}$, let us recall the explicit form of $\sigma_{3\hat{\alpha}}{}^{\hat{\beta}}$, it is

$$\sigma_{3\hat{\alpha}}{}^{\hat{\beta}} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

So let us compute

$$\sigma_{3\hat{\alpha}\hat{\beta}} = C_{\hat{\beta}\hat{1}}\sigma_{3\hat{\alpha}}{}^{\hat{1}} + C_{\hat{\beta}\hat{2}}\sigma_{3\hat{\alpha}}{}^{\hat{2}} = \begin{pmatrix} C_{\hat{1}\hat{1}} & C_{\hat{2}\hat{1}} \\ -C_{\hat{1}\hat{2}} & -C_{\hat{2}\hat{2}} \end{pmatrix},$$

this matrix is symmetric if $\eta(0,2) = -1$ and anti-symmetric if $\eta(0,2) = 1$. We compare this to our result for $\eta(p,d)$ at $p = 2$, $d = 2$

$$\eta(2,2) = -\eta(0,2),$$

which is consistent with this example. Note that this example has not been fully worked through since we have not concluded that $\eta(0.2)$ should be equal to -1, for this some extra computation is required.

Result

To recapitulate, we have the following expression for $\eta(p,d)$ as a function of $\eta(0,d)$

$$\eta(p,d) = \eta(0,d)(-1)^{\frac{1}{2}(p^2+p)}.$$