

Recent advances for integrable long-range spin chains

by

Jules Lamers

Institut de Physique Théorique

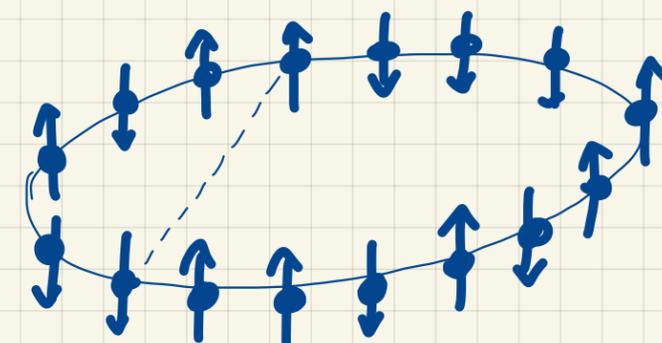
based on

JL
PRB 97 ('18) 214416
arXiv.1801.05728

JL, V Pasquier, D Serban
arXiv 2004.13210

R Klabbers, JL
to appear in CMP
arXiv 2009.14513

and ongoing



Motivation why integrable long-range spin chains

spin chains model magnetic materials based on quantum mechanics

and are **one-dimensional** (nature eg KCuF_3 , laboratory)

long-range interactions

(nature atomic, molecular and optical physics)

applications in string theory

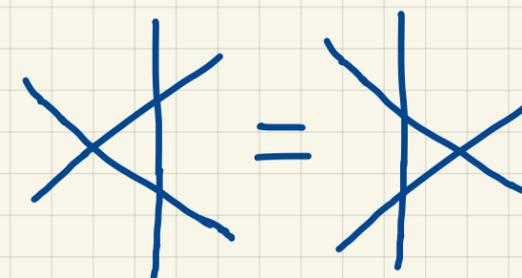
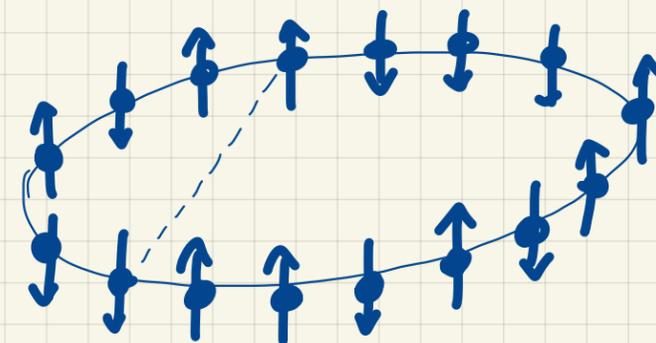
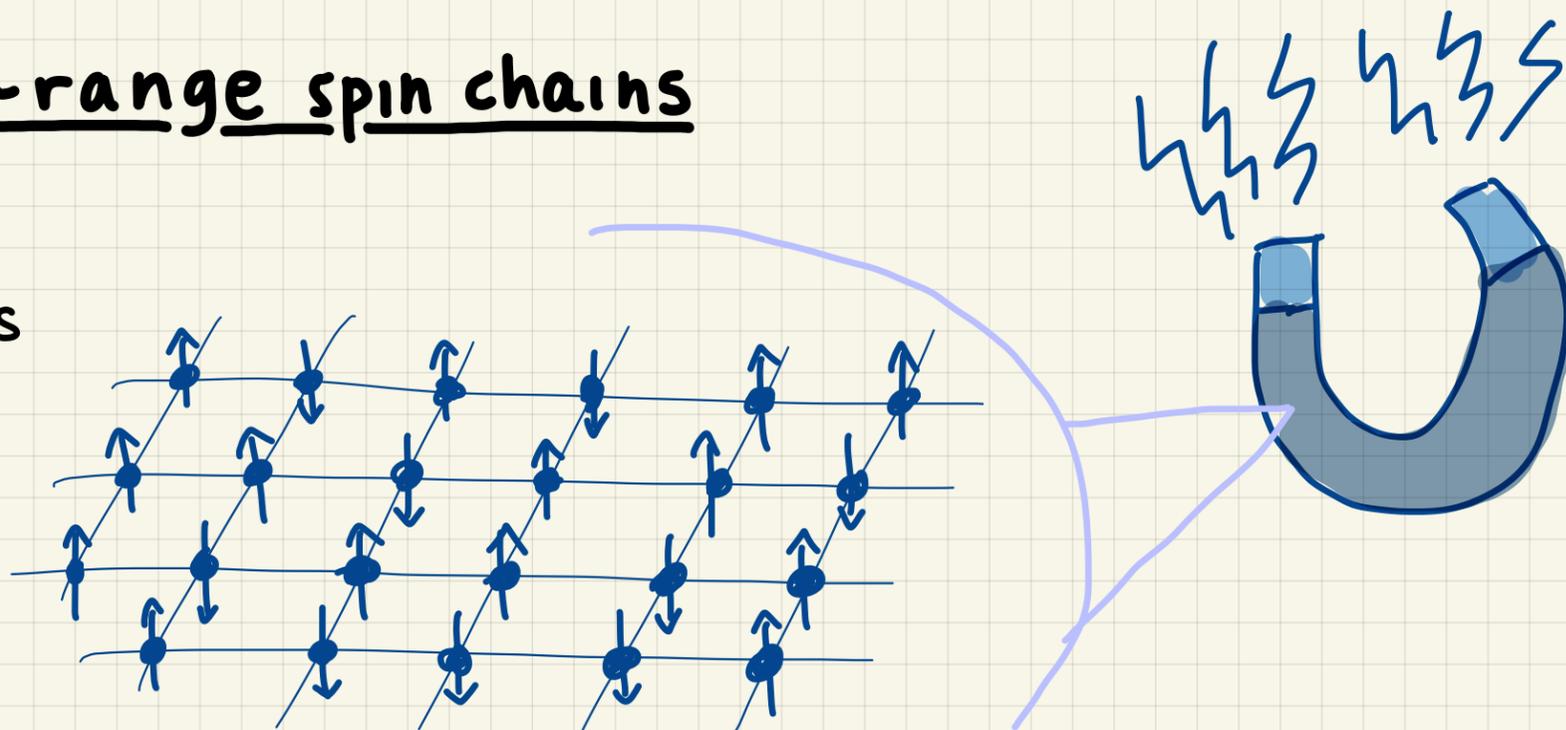
condensed-matter physics

quantum computing

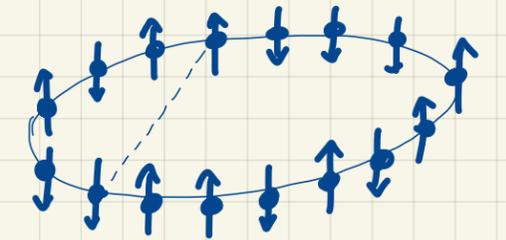
quantum-integrable cases allow one to

study the **effect** of long-range interactions

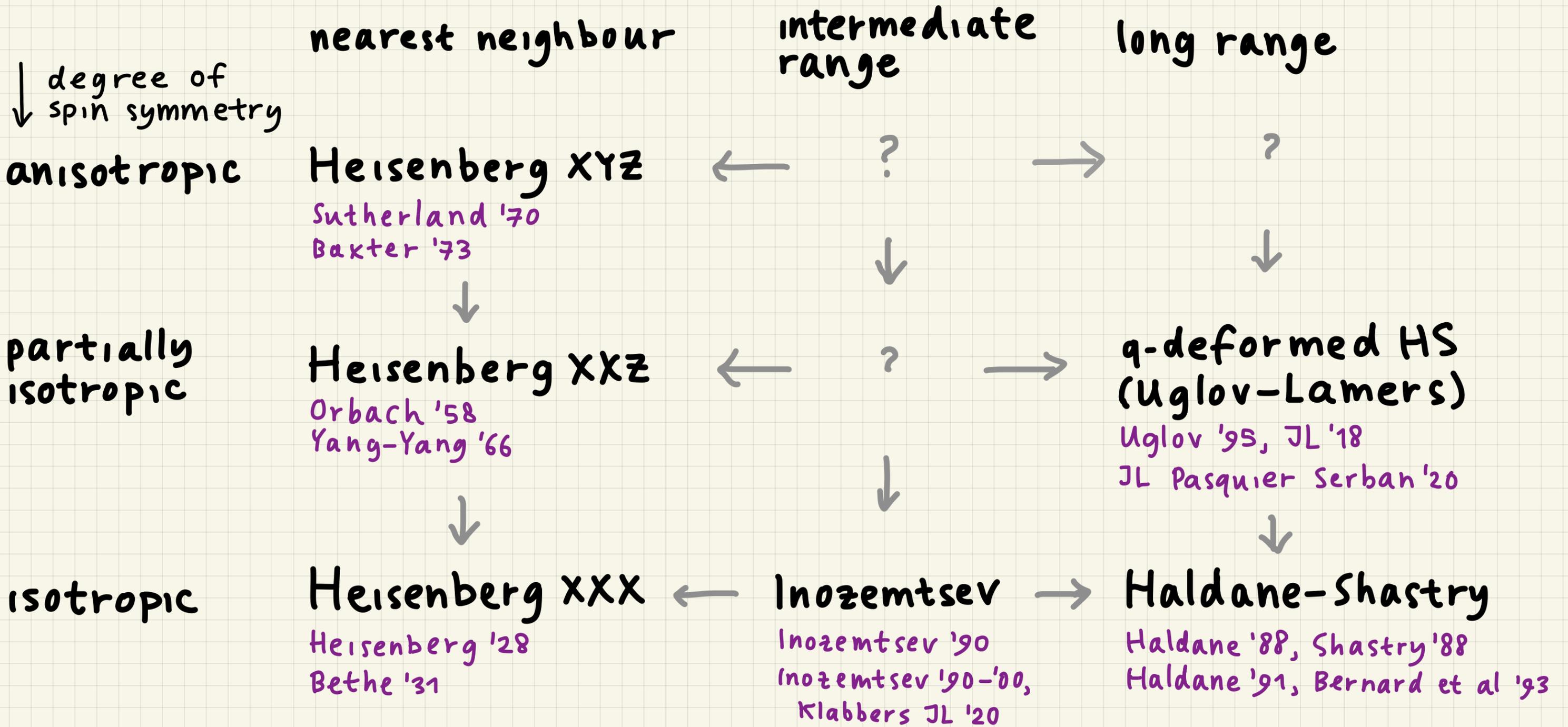
in an **exactly-solvable** setting using quantum algebra



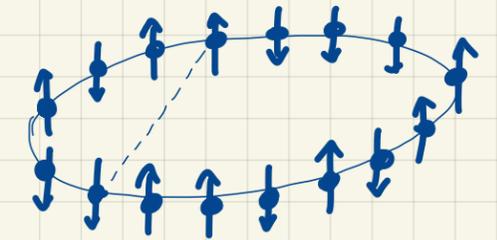
Overview landscape of long-range spin chains



interaction range →



Overview landscape of long-range spin chains



interaction range →

nearest neighbour

intermediate range

long range

↓ degree of spin symmetry

anisotropic

Heisenberg XYZ

Sutherland '70
Baxter '73



?



?



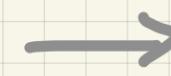
partially isotropic

Heisenberg XXZ

Orbach '58
Yang-Yang '66



?



q-deformed HS (Uglov-Lamers)

Uglov '95, JL '18
JL Pasquier Serban '20



isotropic

Heisenberg XXX

Heisenberg '28
Bethe '31



Inozemtsev

Inozemtsev '90
Inozemtsev '90-'00,
Klabbers JL '20

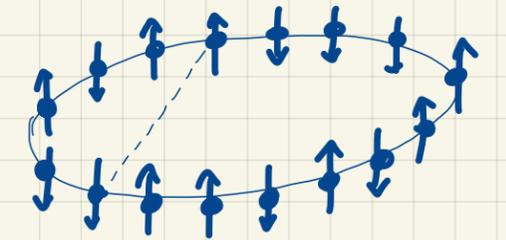


Haldane-Shastry

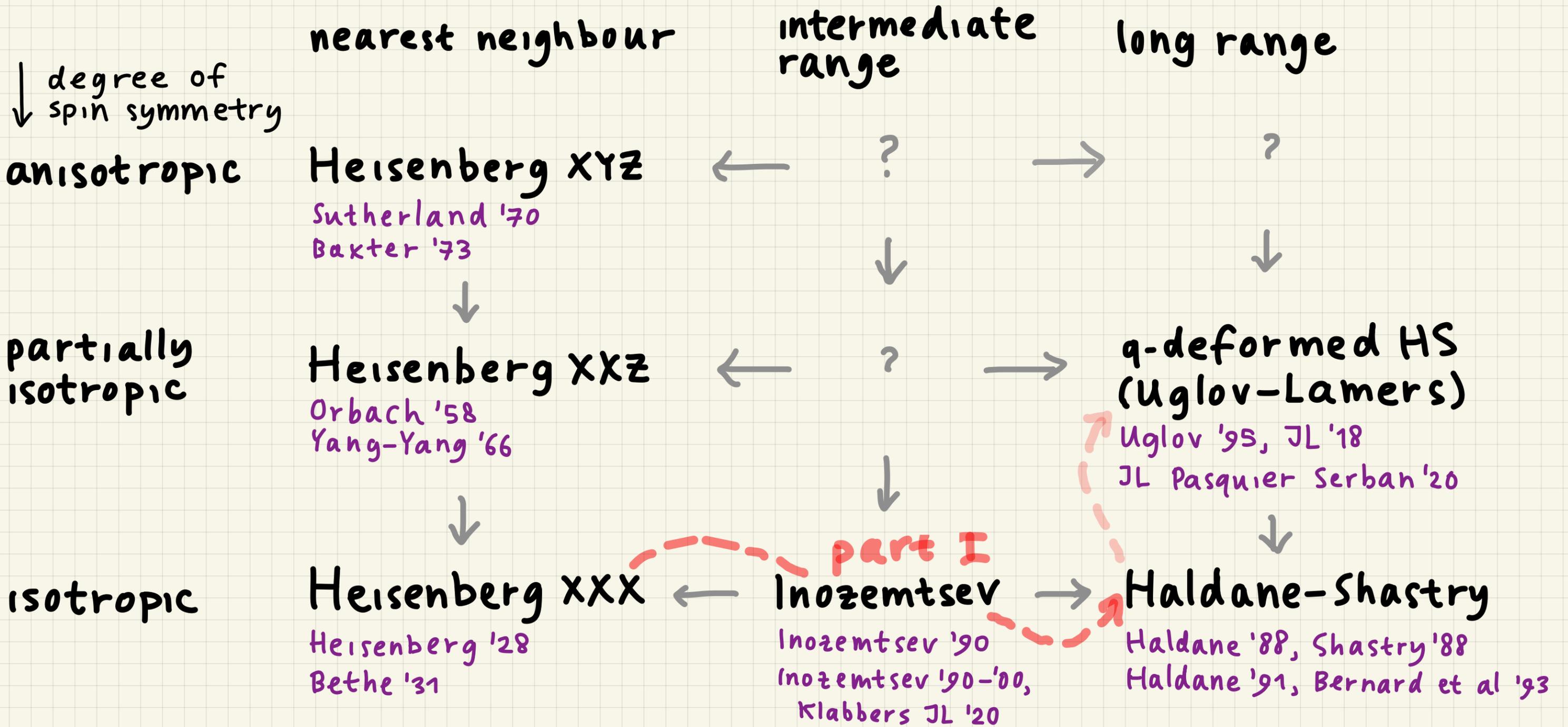
Haldane '88, Shastry '88
Haldane '91, Bernard et al '93

today

Overview landscape of long-range spin chains



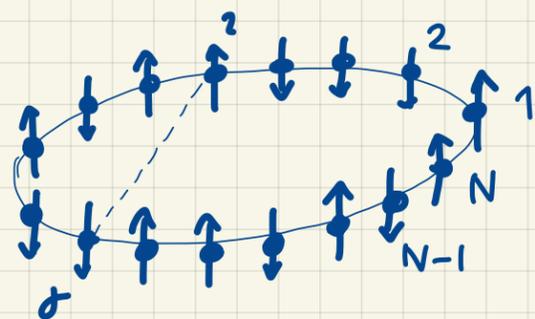
interaction range →



Isotropic level preview

Consider N sites with spin $\frac{1}{2}$

$$H = \sum_{i < j}^N V(i, j) \underbrace{(1 - \vec{\sigma}_i \cdot \vec{\sigma}_j)}_2$$



think

$$\begin{matrix} \delta^2 & \xrightarrow{\kappa \rightarrow 0} & \frac{1}{\sin^2} \\ N \rightarrow \infty \downarrow & & \\ \frac{1}{\sinh^2} & & \end{matrix}$$

Heis XXX '28
nearest neighbour

potential $V_H(i, j) = \delta_{d(i, j), 1}$

up to solving BAE
Bethe '31

alg Bethe ansatz

Yangian structure
Faddeev et al, late '70s

transfer matrix
known

Inozemtsev '90
intermediate range

potential $V_I(i, j) \sim \delta^2(i-j)$
 $(N, \pi/\kappa) \in \mathbb{N}_{\geq 2} \times \mathbb{R}_{> 0}$

up to solving BAE
Inozemtsev '90, '95, '00
Klabbers JL '20

unknown

conjectured,
partial proof
Dittrich Inozemtsev '08

Haldane '88 - Shastry '88
long range

potential $V_{HS}(i, j) = \frac{(\pi/N)^2}{\sin^2(\frac{\pi}{N}(i-j))} = \frac{1}{r^2}$

in closed form **Haldane '91**

degenerate affine Hecke alg
Yangian symmetry
Bernard et al '93

known
Bernard et al '93
Talstra Haldane '95

exact solvability (spectrum)

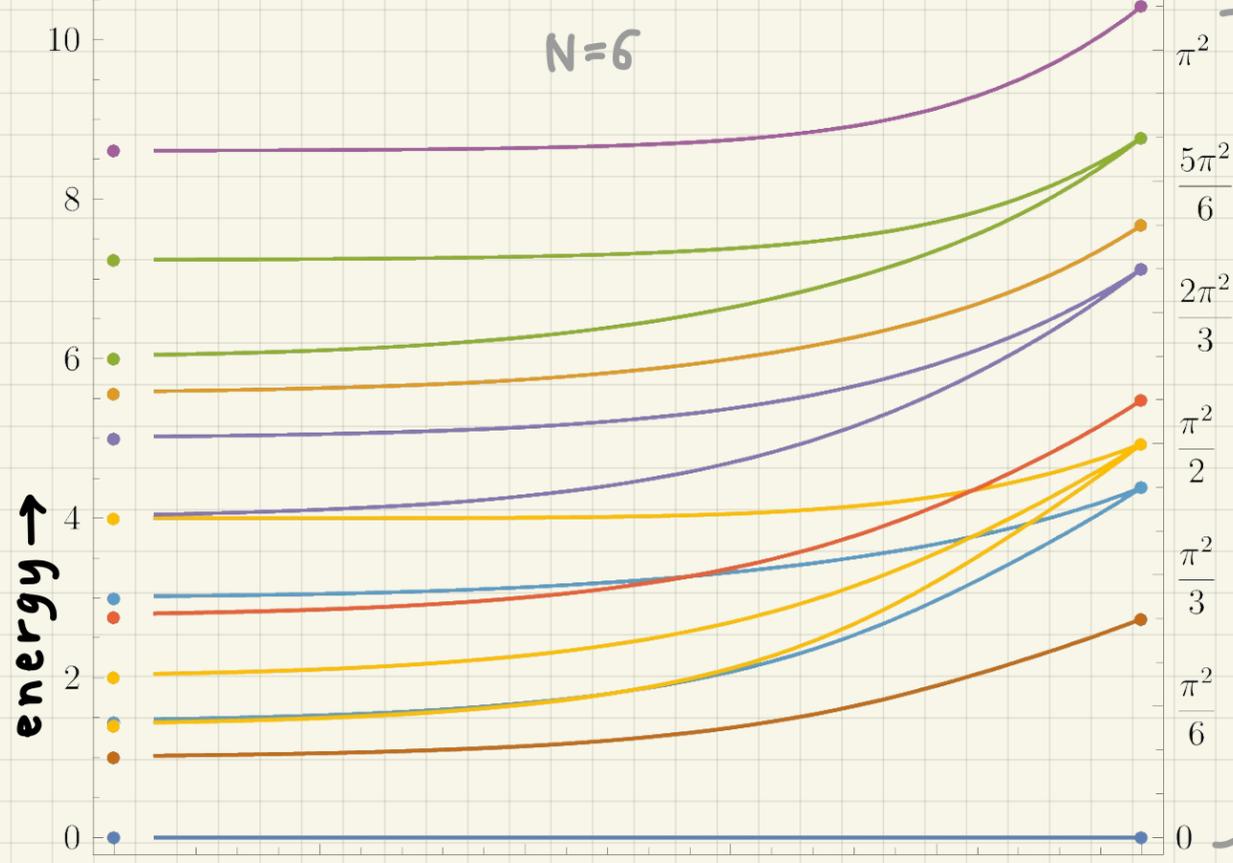
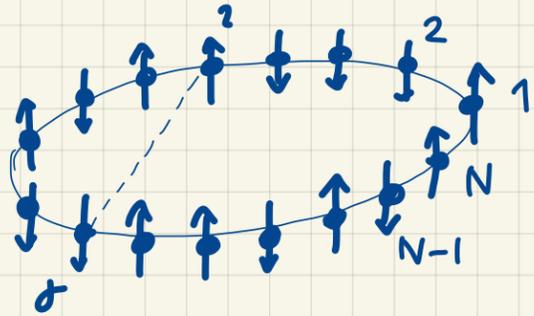
quantum integrab

higher Ham's

Isotropic level preview

Consider N sites with spin $\frac{1}{2}$

$$H = \sum_{i < j}^N V(i, j) \underbrace{(1 - \vec{\sigma}_i \cdot \vec{\sigma}_j)}_2$$



HS spectrum is highly regular
 $E_{HS} \in \frac{2\pi^2}{N^2} \mathbb{Z}_{\geq 0}$,
 highly degenerate
 Haldane '88

Heis XXX '28
 nearest neighbour

up to solving BAE
 Bethe '31

↑ alg Bethe ansatz

Yangian structure
 Faddeev et al, late '70s

↓ transfer matrix
 known

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quantum integrab



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Haldane '88 - Shastry '88
 long range

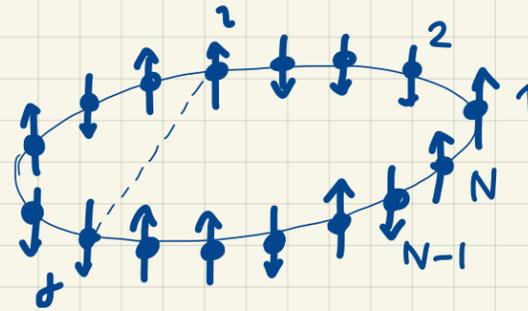
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Isotropic level symmetries

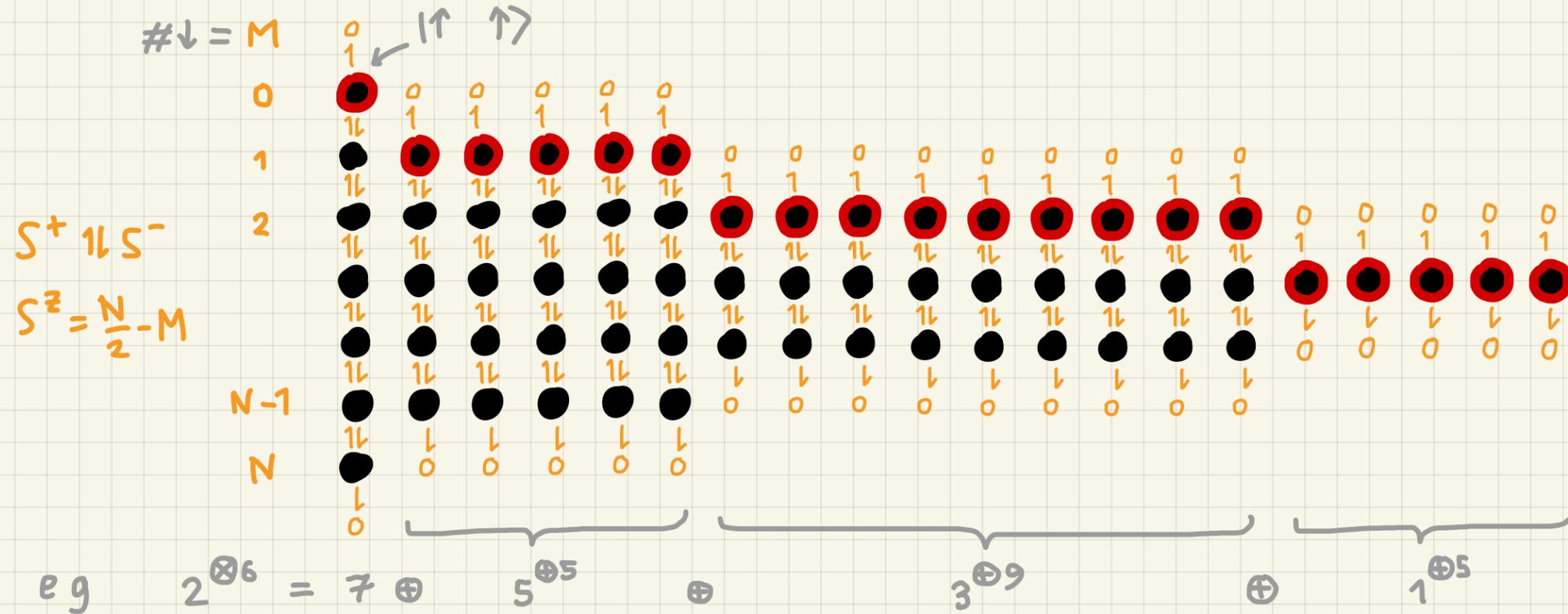
$$H = \sum_{i < j}^N V(r_{ij}) \underbrace{(1 - P_{ij})}_{= (1 - \vec{\sigma}_i \cdot \vec{\sigma}_j) / 2}$$



isotropy $\mathfrak{so}(3)$ allows us to organise $\mathcal{H} = (\mathbb{C}^2)^{\otimes N}$ as

$$S^{\pm} = \sum_{i=1}^N \sigma_i^{\pm} \quad S^z = \sum_{i=1}^N \frac{\sigma_i^z}{2}$$

$$\begin{cases} [S^z, S^{\pm}] = \pm S^{\pm} \\ [S^+, S^-] = 2S^z \end{cases}$$



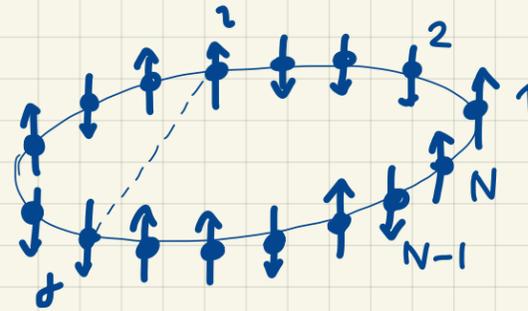
so it remains to find $\mathfrak{so}(3)$ -highest weight vectors at $M \geq 1$

Isotropic level symmetries

$$H = \sum_{i < j}^N V(i, j) (1 - P_{ij})$$

$= (1 - \vec{\sigma}_i \cdot \vec{\sigma}_j) / 2$

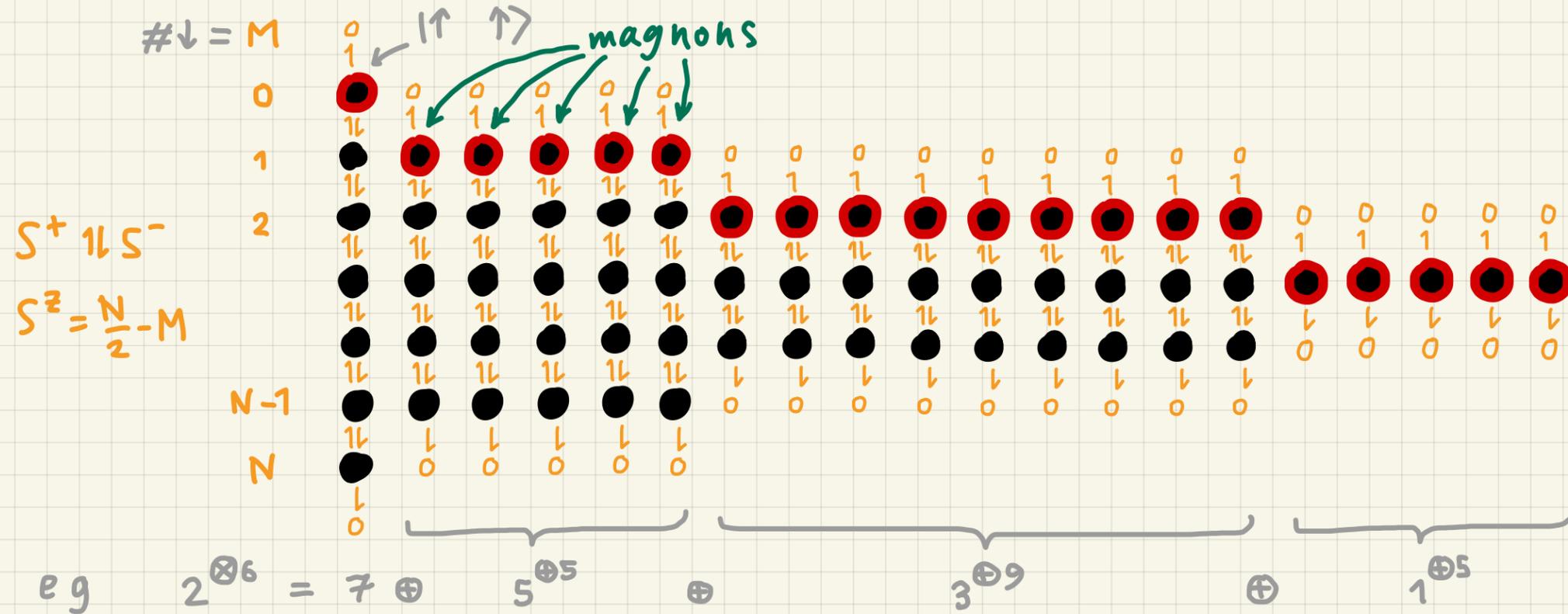
function of dist(i, j)



isotropy global \mathfrak{sl}_2
allows us to organise $\mathcal{H} = (\mathbb{C}^2)^{\otimes N}$ as

$$S^{\pm} = \sum_{i=1}^N \sigma_i^{\pm} \quad S^z = \sum_{i=1}^N \frac{1}{2} \sigma_i^z$$

$$\begin{cases} [S^z, S^{\pm}] = \pm S^{\pm} \\ [S^+, S^-] = 2S^z \end{cases}$$



so it remains to find \mathfrak{sl}_2 -highest weight vectors at $M \geq 2$

homogeneity translations
fixes $M=1$

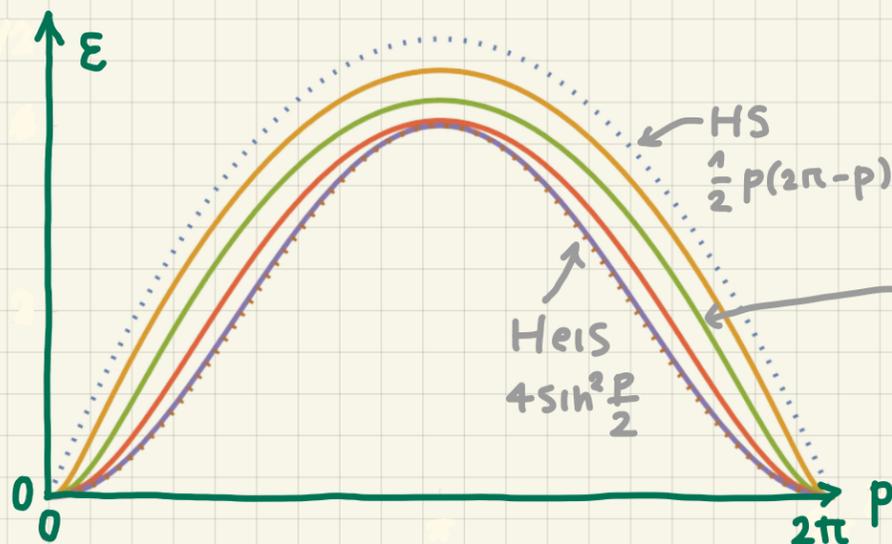
magnons

$$\sum_{n=1}^N e^{iPn} \sigma_n^- |\uparrow \uparrow \rangle$$

momentum

$$P = \frac{2\pi}{N} I$$

$(1 \leq I \leq N-1)$



$I_{no} \quad (2\pi, i\pi\kappa) \quad \vec{S}' = -\vec{S}$

$$\sim \vec{S}(p) - (\vec{S}(p) + \text{cst } p)^2$$

$$\sim \frac{\vec{S}''(p)}{\vec{S}(p)} + \text{cst}$$

Isotropic level $M=2$

Haldane et al '91-2
Klabbers JL '20

Heis XXX

energy

$$E_H = \varepsilon_H(p_1) + \varepsilon_H(p_2)$$

dispersion

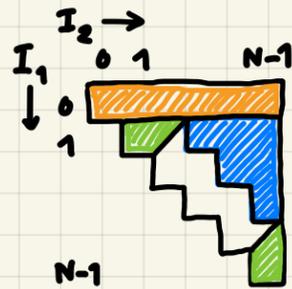
$$\varepsilon_H(p) = 4 \sin^2 \frac{p}{2}$$

quasimomenta

$$\begin{cases} p_{1,2} = \frac{2\pi}{N} I_{1,2} \pm \theta_H(p_1, p_2) \\ 0 \leq I_1, I_2 \leq N-1 \end{cases}$$

Bethe integers

$$0 \leq I_1, I_2 \leq N-1$$



$$I_1 = 0 \quad \# = N$$

$$\theta_H = 0$$



$$E_H = \varepsilon_H(p_2) \text{ as at } M=1$$

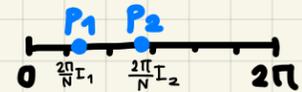
\mathfrak{sl}_2 -descendant (S^1)

\mathfrak{sl}_2 -hw

$$I_1 > 0, I_2 > I_1 + 1$$

$$\# = \binom{N-2}{2}$$

$$\exists! \theta_H \in (0, \frac{\pi}{N})$$



$$E_H \approx \varepsilon_H(\frac{2\pi}{N} I_1) + \varepsilon_H(\frac{2\pi}{N} I_2)$$

scattering

$$I_1 > 0, \text{ suitable } I_2 \in \{I_1, I_1 + 1\}$$

$$\# = N-3$$

$$\exists! \theta_H \in \frac{2\pi}{N} \frac{I_2 - I_1}{2} + i\mathbb{R} > 0 \quad N \text{ suff low}$$



$$E_H \approx \frac{1}{2} \varepsilon_H(\frac{2\pi}{N} (I_1 + I_2))$$

bound state

Inozemtsev

on shell

$$E_I = \varepsilon_I(p_1) + \varepsilon_I(p_2) - \tilde{U}(p_1, p_2)$$

←

$$\varepsilon_I(p) \sim \bar{\mathcal{G}}(p) - (\bar{\mathcal{S}}(p) + \text{cst } p)^2$$

←

$$\begin{cases} p_{1,2} = \frac{2\pi}{N} I_{1,2} \pm \theta_I(p_1, p_2) \\ 0 \leq I_1, I_2 \leq N-1 \end{cases}$$

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$$0 \leq I_1, I_2 \leq N-1$$

Haldane-Shastry

$$E_{HS} = \varepsilon_{HS}(p_1) + \varepsilon_{HS}(p_2)$$

$$\varepsilon_{HS}(p) = \frac{1}{2} p (2\pi - p)$$

$$\begin{cases} p_{1,2} = \frac{2\pi}{N} I_{1,2} \pm \theta_{HS}(p_1, p_2) \\ 0 \leq I_1, I_2 \leq N-1 \end{cases}$$

$$0 \leq I_1, I_2 \leq N-1$$

Isotropic level $M=2$

Haldane et al '91-2
Klabbers JL '20

Heis XXX

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dispersion

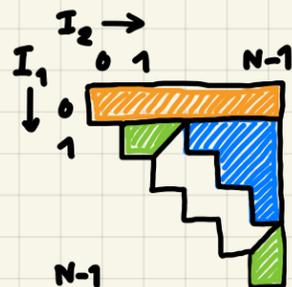
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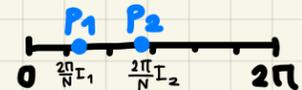
\mathcal{H}_2 -descendant (s^-)

\mathcal{H}_2 -hw

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$\# = \binom{N-2}{2}$

$$\exists! \theta_H \in (0, \frac{\pi}{N})$$



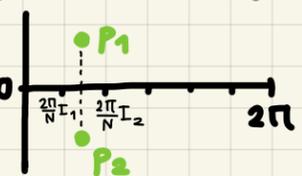
$$E_H \approx \varepsilon_H(\frac{2\pi}{N} I_1) + \varepsilon_H(\frac{2\pi}{N} I_2)$$

scattering

$$I_1 > 0, \text{ suitable } I_2 \in \{I_1, I_1 + 1\}$$

$\# = N-3$

$$\exists! \theta_H \in \frac{2\pi}{N} \frac{I_2 - I_1}{2} + i\mathbb{R}, > 0 \quad N \text{ suff low}$$



$$E_H \approx \frac{1}{2} \varepsilon_H(\frac{2\pi}{N} (I_1 + I_2))$$

bound state

Inozemtsev

on shell

$$E_I = \varepsilon_I(p_1) + \varepsilon_I(p_2) - \tilde{U}(p_1, p_2)$$

←

$$\varepsilon_I(p) \sim \bar{\varphi}(p) - (\bar{\zeta}(p) + \text{cst } p)^2$$

←

$$\begin{cases} p_{1,2} = \frac{2\pi}{N} I_{1,2} \pm \theta_I(p_1, p_2) \\ 0 \leq I_1, I_2 \leq N-1 \end{cases}$$

←

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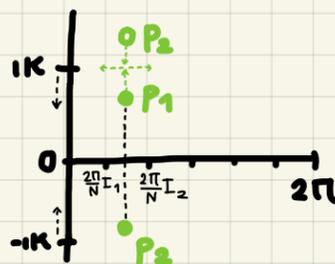
\mathcal{H}_2 -descendant (s^-)

$$\exists! \theta_I \in (0, \theta_H)$$



$$E_I \approx \varepsilon_I(\frac{2\pi}{N} I_1) + \varepsilon_I(\frac{2\pi}{N} I_2)$$

scattering



$$E_I > E_H$$

Isotropic level $M=2$

Haldane et al '91-2
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Heis XXX

energy
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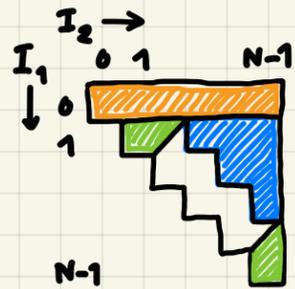
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$$\theta_H = 0$$

$$E_H = \varepsilon_H(p_2) \text{ as at } M=1$$

\mathfrak{sl}_2 -descendant (s^-)

$$\theta_I = 0$$

$$E_I = \varepsilon_I(p_2) \text{ as at } M=1$$

\mathfrak{sl}_2 -descendant (s^-)

$$\theta_{HS} = 0$$

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\mathfrak{sl}_2 -descendant (s^-)

\mathfrak{sl}_2 -hw

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scattering

$$\exists! \theta_I \in (0, \theta_H)$$

$$E_I \approx \varepsilon_I(\frac{2\pi}{N} I_1) + \varepsilon_I(\frac{2\pi}{N} I_2)$$

scattering

$$\theta_{HS} = 0$$

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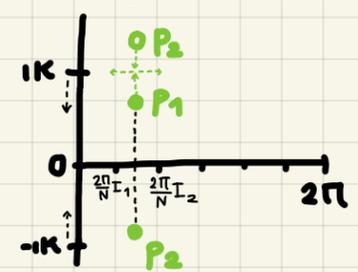
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bound state

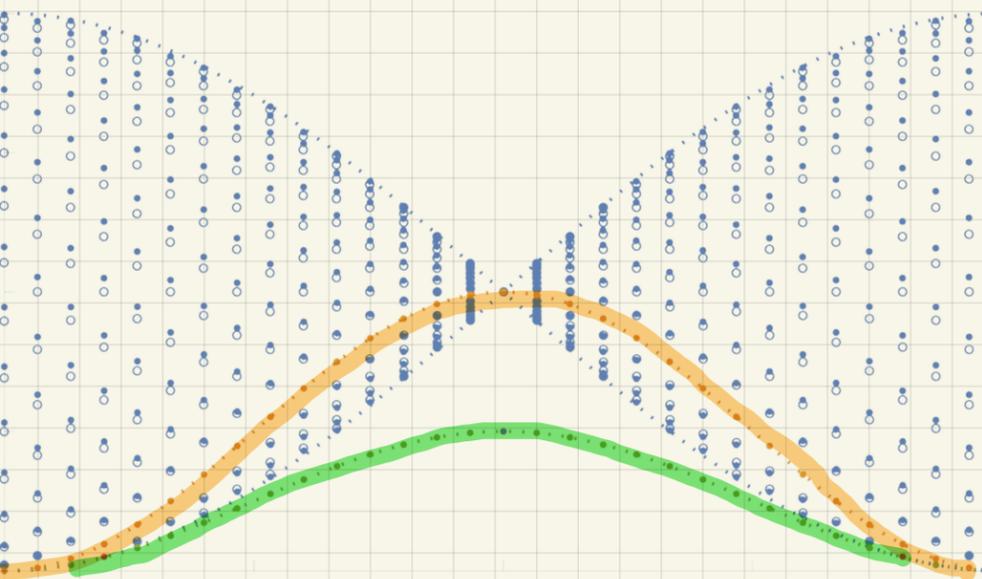
$$E_I > E_H$$

$$\theta_{HS} = -\frac{2\pi}{N} I_1$$

$$E_{HS} = \varepsilon_{HS}(p_2) \text{ as at } M=1$$

Isotropic level $M=2$

Heis XXX



$I_1=0 \quad \# = N$

$\theta_H=0$

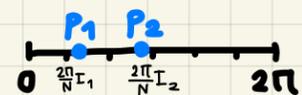


$E_H = \varepsilon_H(p_2)$ as at $M=1$
 \mathfrak{sl}_2 -descendant (s^-)

\mathfrak{sl}_2 -hw

$I_1 > 0, I_2 > I_1 + 1$
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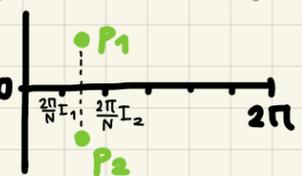
$\exists! \theta_H \in (0, \frac{\pi}{N})$



$E_H \approx \varepsilon_H(\frac{2\pi}{N} I_1) + \varepsilon_H(\frac{2\pi}{N} I_2)$
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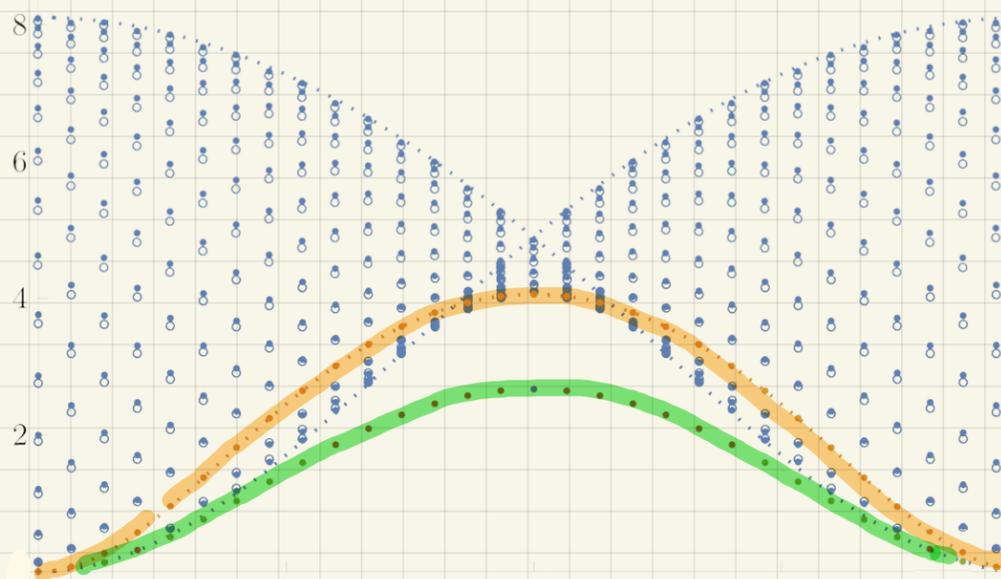
$I_1 > 0$, suitable
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$E_H \approx \frac{1}{2} \varepsilon_H(\frac{2\pi}{N} (I_1 + I_2))$
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Inozemtsev

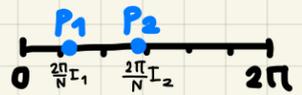


$\theta_I=0$

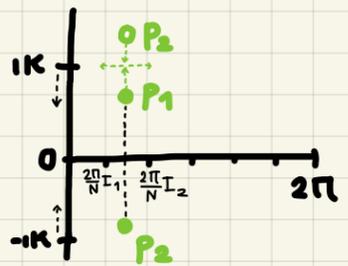


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 \mathfrak{sl}_2 -descendant (s^-)

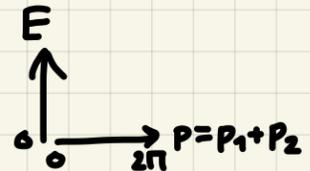
$\exists! \theta_I \in (0, \theta_H)$



$E_I \approx \varepsilon_I(\frac{2\pi}{N} I_1) + \varepsilon_I(\frac{2\pi}{N} I_2)$
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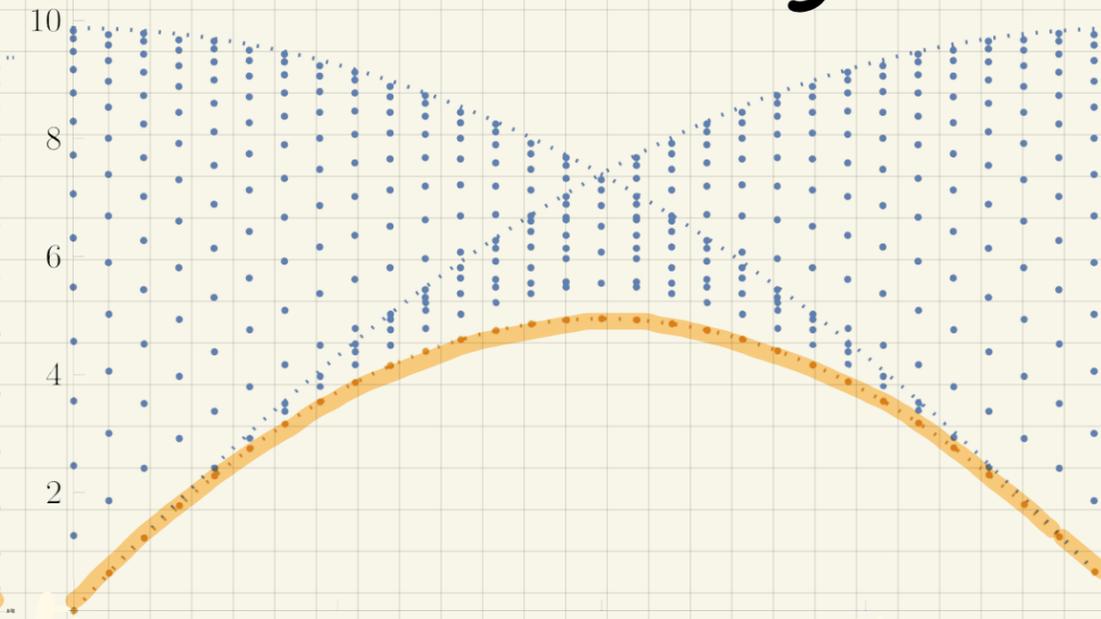


$E_I > E_H$



Haldane-Shastry

Haldane et al '91-2
 Klabbers JL '20

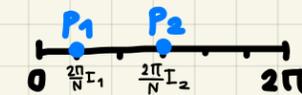


$\theta_{HS}=0$



$E_{HS} = \varepsilon_{HS}(p_2)$ as at $M=1$
 \mathfrak{sl}_2 -descendant (s^-)

$\theta_{HS}=0$



$E_{HS} = \varepsilon_{HS}(\frac{2\pi}{N} I_1) + \varepsilon_{HS}(\frac{2\pi}{N} I_2)$

$\theta_{HS} = -\frac{2\pi}{N} I_1$



$E_{HS} = \varepsilon_{HS}(p_2)$ as at $M=1$

Isotropic level hidden symmetry of HS

$$H_{HS} = \sum_{i < j}^N \frac{(\pi/N)^2}{\sin^2(\frac{\pi}{N}(i-j))} (1 - P_{ij})$$

Ha et al '92
Bernard et al '93

Yangian symmetry

$$\mathfrak{sl}_2 \quad S^{\pm} = \sum_{i=1}^N \sigma_i^{\pm} \quad S^z = \sum_{i=1}^N \frac{\sigma_i^z}{2}$$

$$\begin{cases} [S^z, S^{\pm}] = \pm S^{\pm} \\ [S^+, S^-] = 2S^z \end{cases}$$

\cap
 $Y\mathfrak{sl}_2$
(∞ -dim)

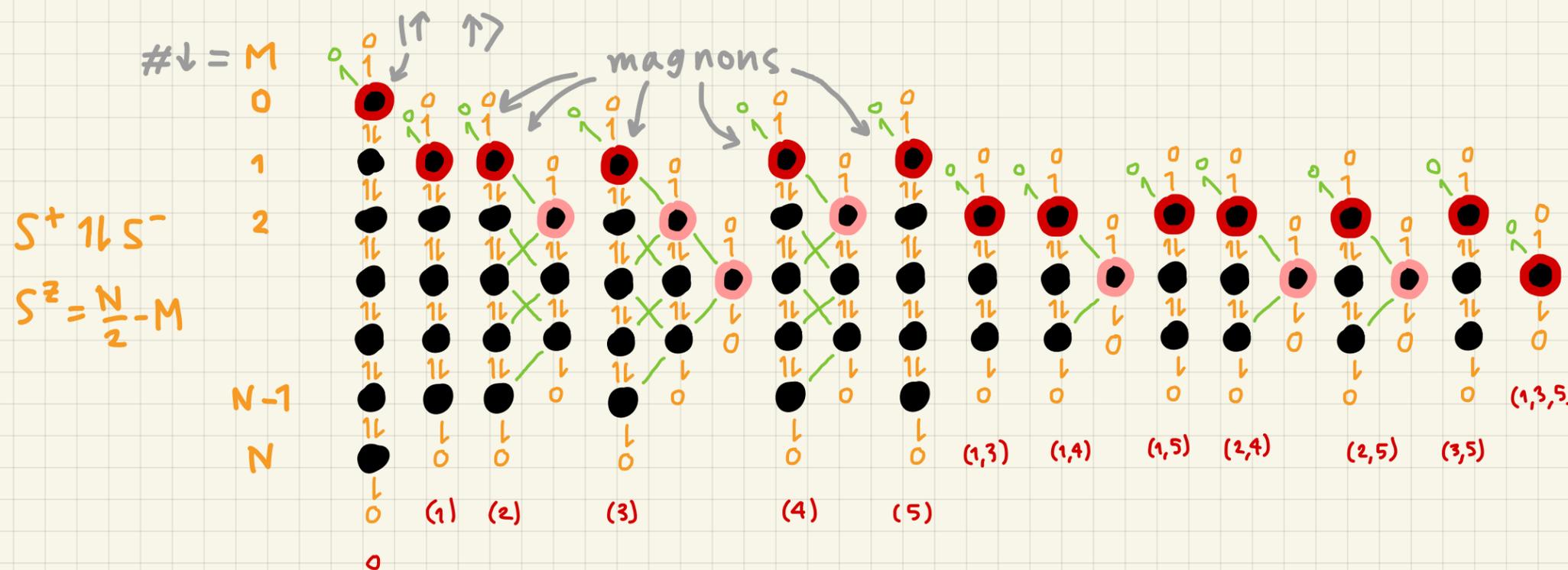
$$Q^{\pm} = \pm \frac{1}{2} \sum_{i < j}^N \cot(\frac{\pi}{N}(i-j)) (\sigma_i^z \sigma_j^{\pm} - \sigma_i^{\pm} \sigma_j^z)$$

$$\begin{cases} [S^z, Q^{\pm}] = \pm Q^{\pm} \\ [S^{\pm}, Q^{\mp}] = \pm 2Q^z \\ [S^{\pm}, Q^z] = \mp Q^{\pm} \end{cases}$$

& Serre-like relation

$$Q^z = \frac{1}{2} \sum_{i < j}^N \cot(\frac{\pi}{N}(i-j)) (\sigma_i^+ \sigma_j^- - \sigma_i^- \sigma_j^+)$$

allows us to organise $\mathcal{H} = (\mathbb{C}^2)^{\otimes N}$ as



so it remains to find
 $Y\mathfrak{sl}_2$ -highest weight vectors
at $M \geq \star 2$

labelled by **motifs**

$$1 \leq \mu_m \leq N-1$$

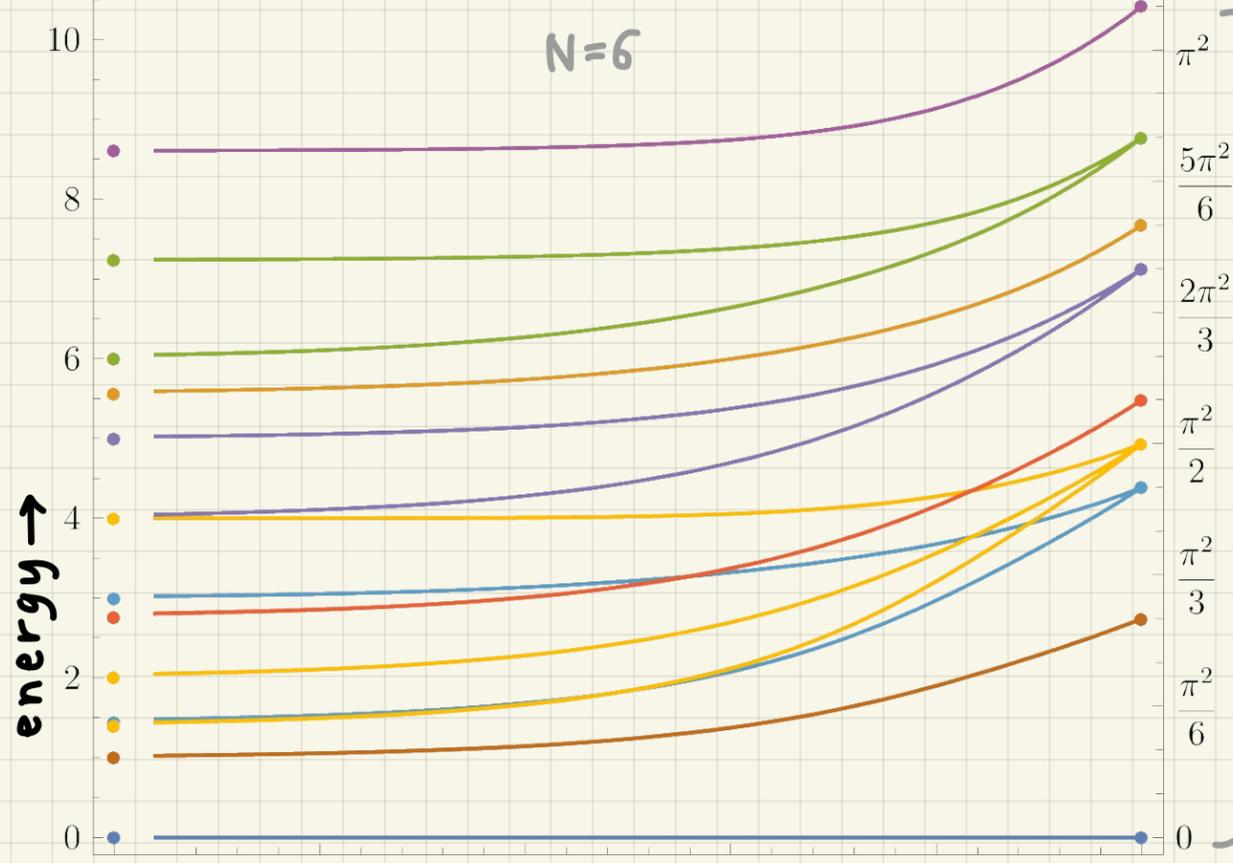
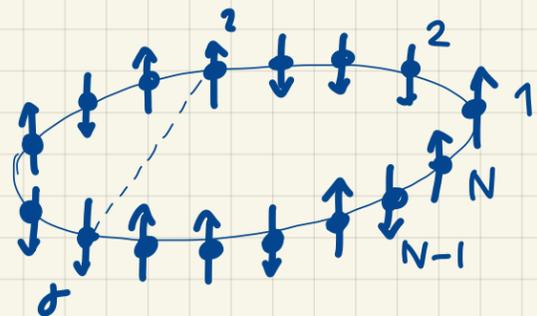
$$\mu_{m+1} > \mu_m + 1$$

$$\text{s.t. } P_m = \frac{2\pi}{N} \mu_m$$

Isotropic level summary

Consider N sites with spin $\frac{1}{2}$

$$H = \sum_{i < j}^N V(i, j) \underbrace{(1 - \vec{\sigma}_i \cdot \vec{\sigma}_j)}_2$$



HS spectrum is highly regular
 $E_{HS} \in \frac{2\pi^2}{N^2} \mathbb{Z}_{\geq 0}$,
 highly degenerate
 Haldane '88

Heis XXX '28
 nearest neighbour

up to solving BAE
 Bethe '31

↑ alg Bethe ansatz

Yangian structure
 Faddeev et al, late '70s

↓ transfer matrix
 known

exact solvability (spectrum)



quantum integrab



higher Ham's

Inozemtsev '90
 intermediate range

up to solving BAE
 Inozemtsev '90, '95, '00
 Klabbers JL '20

unknown

conjectured, partial proof
 Dittrich Inozemtsev '08

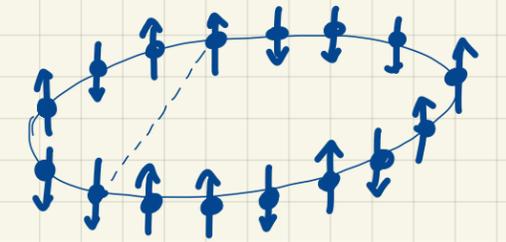
Haldane '88 - Shastry '88
 long range

in closed form Haldane '91

↑
 degenerate affine Hecke alg
 Yangian symmetry
 Bernard et al '93

↓ known
 Bernard et al '93
 Talstra Haldane '95

Overview landscape of long-range spin chains



interaction range →

nearest neighbour

intermediate range

long range

↓ degree of spin symmetry

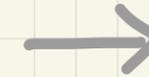
anisotropic

Heisenberg XYZ

Sutherland '70
Baxter '73



?



?



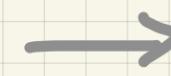
partially isotropic

Heisenberg XXZ

Orbach '58
Yang-Yang '66



?



q-deformed HS
(Uglov-Lamers)

Uglov '95, JL '18
JL Pasquier Serban '20



part II



isotropic

Heisenberg XXX

Heisenberg '28
Bethe '31



Inozemtsev

Inozemtsev '90
Inozemtsev '90-'00,
Klabbers JL '20



Haldane-Shastry

Haldane '88, Shastry '88
Haldane '91, Bernard et al '93

Isotropic level algebraic structure of HS

Bernard et al '93
Talstra Haldane '95

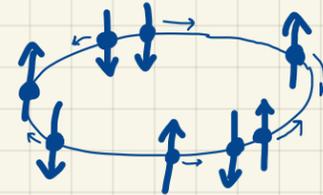
degenerate
affine Hecke
algebra



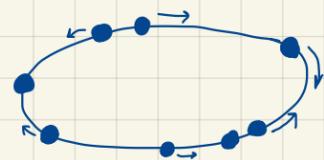
N bosons
with spin $\frac{1}{2}$

trig spin-Cal-Sut

- commuting Hams
- Yangian symm

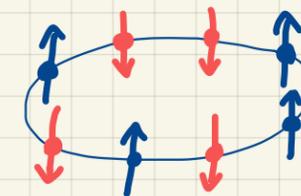
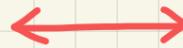
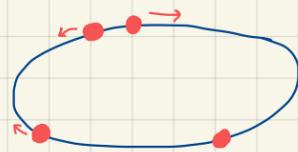


N bosons



trig Cal-Sut

- commuting Hams
- exact eigenfunctions Jacks



Haldane-Shastry

- commuting Hams
- Yangian symm
- exact eigenvectors



freezing

semiclass lim
to class equilb

Isotropic level algebraic structure of HS

Bernard et al '93
Talstra Haldane '95

permutations
Dunkl operators

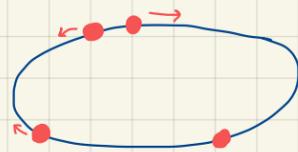
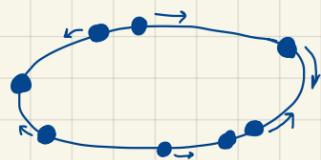
degenerate
affine Hecke
algebra

$k = \alpha' \in \mathbb{C}$

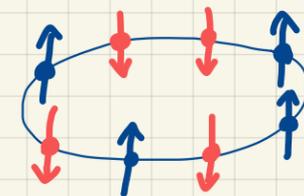
N bosons
with spin $\frac{1}{2}$
 $\text{Sym}^N \mathbb{C}^2 [e^{ix}]$
 $= (\mathbb{C}^2)^{\otimes N} \otimes_{S_N} \mathbb{C} [e^{ix_1}, \dots, e^{ix_N}]$

$k = \alpha' \in \mathbb{C}$
reduced
coupling

N bosons
 $\text{Sym}^N \mathbb{C} [e^{ix}]$
 $= \mathbb{C} [e^{ix_1}, \dots, e^{ix_N}]^{S_N}$



$N^* = M$
 $k^* = 2$
 $x_m^* = \frac{\pi}{N} n_m$



trig Cal-Sut

- commuting Hams
- exact eigenfunctions Jacks

Haldane-Shastry

- commuting Hams
- Yangian symm
- exact eigenvectors

$$H_{HS} = \sum_{i < j} \frac{(\pi/N)^2 (1 - P_{ij})}{\sin(\frac{\pi}{N}(i-j))^2}$$

$$\mathbb{F}(n_1, \dots, n_M) = \tilde{\mathbb{F}}(\omega^{n_1}, \dots, \omega^{n_M}), \quad \omega = e^{2\pi i/N}$$

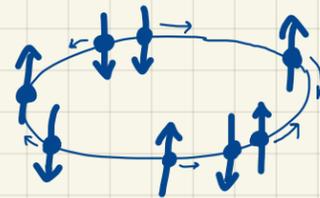
$$\tilde{\mathbb{F}}_{\mu}(z_1, \dots, z_M) = \underbrace{\prod_{m < m'} (z_m - z_{m'})^2}_{\text{Vandermonde squared}} \times \underbrace{P_{\lambda(\mu)}^{(\frac{1}{2})}(z_1, \dots, z_M)}_{\text{Jack polynomial (zonal spherical)}}$$

Vandermonde
squared

Jack polynomial
(zonal spherical)

trig spin-Cal-Sut

- commuting Hams
- Yangian symm



freezing

semiclass lim
to class equilb $x_j^* = \frac{2\pi}{N} j$

$$\tilde{H}_{tCS} = -\frac{1}{2} \sum_{j=1}^N \partial_{x_j}^2 + \sum_{i < j} \frac{k(k - P_{ij})}{4 \sin((x_i - x_j)/2)^2}$$

$\exists Q^x$ Dunkls instead of inhomogenities

Partially isotropic level algebraic structure

Bernard et al '93

Uglov '95, JL '18

JL Pasquier Serban '20

partial isotropy $[S^z, H] = 0$

$q \in \mathbb{C}$ q -permutations T_i
 q -Dunkl operators Y_i

affine Hecke algebra

$p = q^{2/k} \in \mathbb{C}$
 $t_{\text{Macd}} = t_{\text{Macd}}^{1/k}$

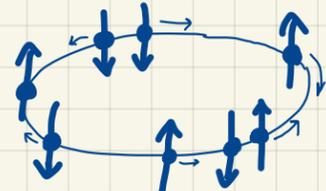
relativistic N (q -)bosons
 $\text{Sym}_q^N \mathbb{C}[e^{ix}]$
 $= \mathbb{C}[e^{ix_1}, \dots, e^{ix_N}]_{H_N}$
 $= \mathbb{C}[e^{ix_1}, \dots, e^{ix_N}]_{S_N}$

$p = q^{2/k} \in \mathbb{C}$
 $t_{\text{Macd}} = t_{\text{Macd}}^{1/k}$

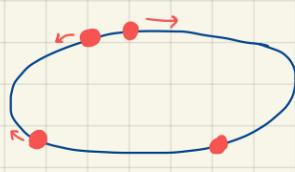
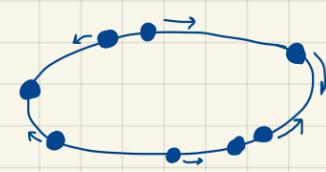
N q -bosons with spin $\frac{1}{2}$
 $\text{Sym}_q^N \mathbb{C}^2[e^{ix}]$
 $= (\mathbb{C}^2)^{\otimes N} \otimes_{H_N} \mathbb{C}[e^{ix_1}, \dots, e^{ix_N}]$

trig spin-Ruijsenaars-Macdonald

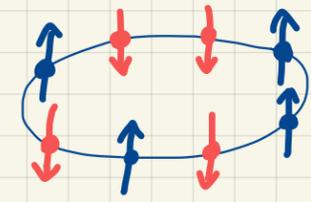
- commuting Hams
 - quantum-loop symm
- explicit q -Dunkls instead of inhomogenities



freezing
 semiclassical lim to class equil
 $k \rightarrow \infty$
 $x_j^* = \frac{2\pi}{N} j$



$N^* = M$
 $k^* = 2$
 $x_m^* = \frac{\pi}{N} n_m$



trig Ruijsenaars-Macdonald

- commuting Hams explicit
- exact eigenfunctions Macdonald

q -deformed Haldane-Shastry

- commuting Hams
- quantum-loop symm
- exact eigenvectors $w = e^{2\pi i / N}$

$$\tilde{\Psi}_\mu(z_1, \dots, z_M) = \underbrace{\prod_{m < m'}^M (qz_m - q^{-1}z_{m'}) \times (q^{-1}z_m - qz_{m'})}_{\text{'symmetric square' of } q\text{-Vandermonde}} \times \underbrace{P_{\lambda(\mu)}(z_1, \dots, z_M, q^{1/2}, q^2)}_{\text{Macdonald polynomial (quantum zonal spherical)}}$$

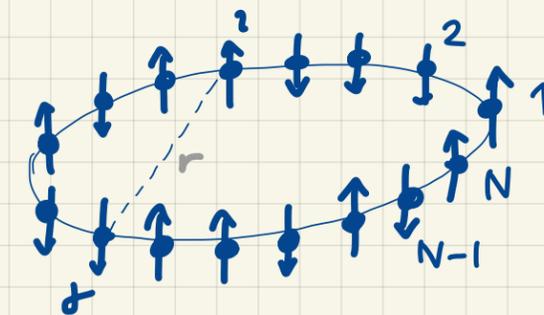
Partially isotropic level summary

$$H = \sum_{i < j}^N V(i, j) S_{[i, j]}$$

partially isotropic $[S^z, H] = 0$
not quite homogeneous

$$S_{[i, j]} = \begin{array}{|c|} \hline \text{---} \\ \hline \end{array} \begin{array}{|c|} \hline \text{---} \\ \hline \end{array}$$

z_i, z_{i+1}, z_j



Heis XXZ Orbach '58
nearest neighbour

$$V_{\text{Heis}}(i, j) = \delta_{d(i, j), 1}$$

unknown
intermediate

conjecture
Klabbers JL '20

Uglov '95 - JL '18
long range

$$V_{\text{UL}}(i, j) = \frac{1}{r_+ r_-}$$

potential

exact
solvability
(spectrum)

up to solving BAE
Yang-Yang '66

unknown

in closed form JL et al '20
via connection to
trig Ruijsenaars-Macdonald



↑ alg Bethe ansatz



quantum
integrals

quantum-loop structure
Faddeev et al, late '70s

unknown

affine Hecke alg
quantum-loop symmetry
Bernard et al '93



↓ transfer matrix



higher
Ham's

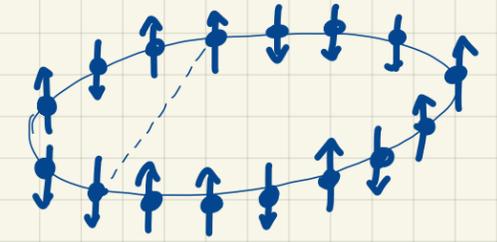
known

unknown

known

JL et al '20

Outlook landscape of long-range spin chains



interaction range →

↓ degree of spin symmetry

anisotropic

partially isotropic

isotropic

nearest neighbour

intermediate range

long range

Heisenberg XYZ

Sutherland '70
Baxter '73

Heisenberg XXZ

Orbach '58
Yang-Yang '66

Heisenberg XXX

Heisenberg '28
Bethe '31

Inozemtsev

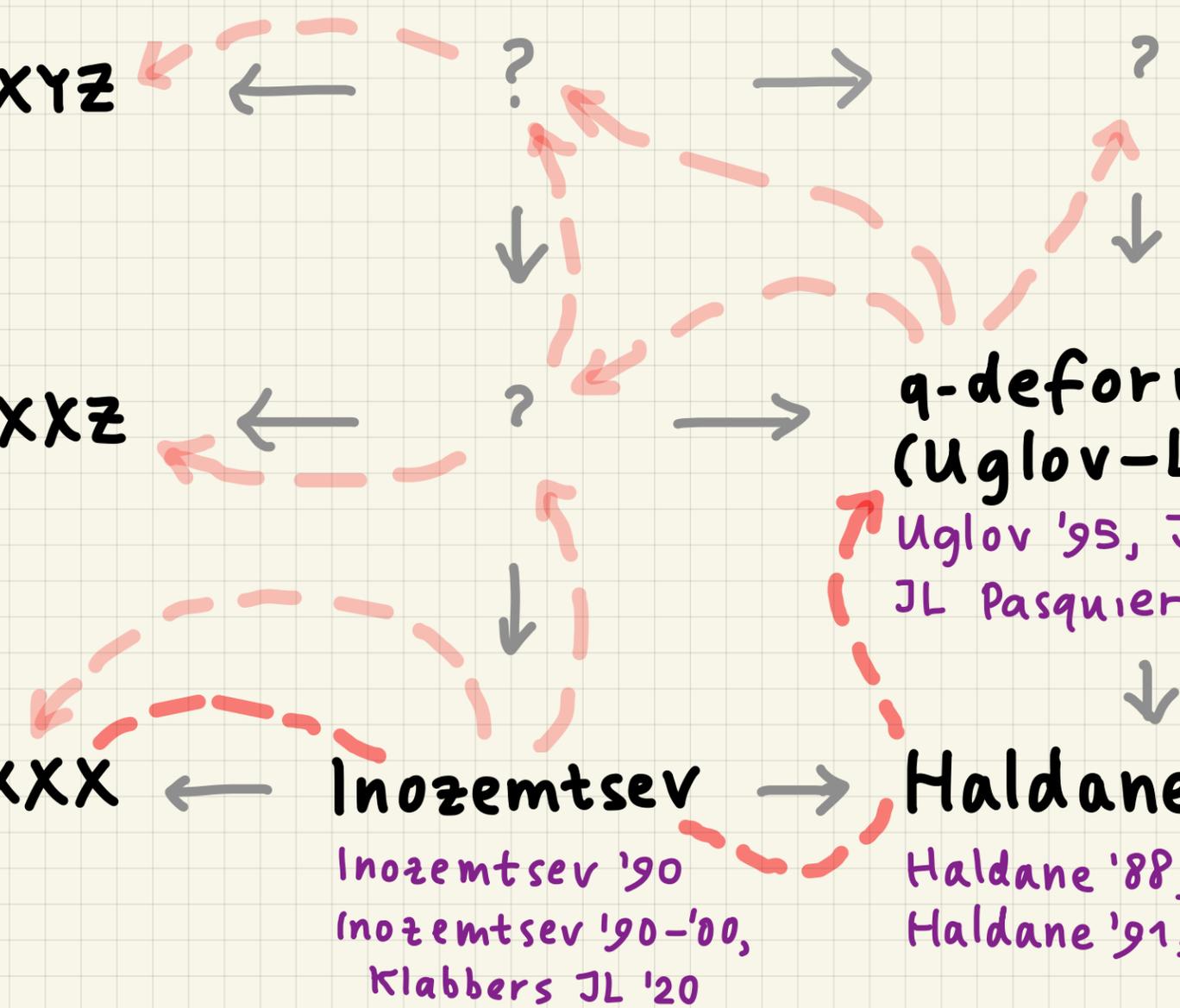
Inozemtsev '90
Inozemtsev '90-'00,
Klabbers JL '20

q-deformed HS
(Uglov-Lamers)

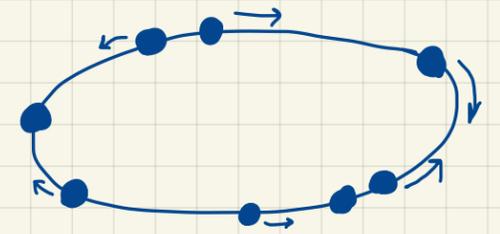
Uglov '95, JL '18
JL Pasquier Serban '20

Haldane-Shastry

Haldane '88, Shastry '88
Haldane '91, Bernard et al '93



Outlook landscape of quantum many-body systems



interaction range →

nearest neighbour
contact (positions)

intermediate
range
elliptic (positions)

long range
trig (positions)

(?)
elliptic (momenta)

?

'DELL'

?

relativistic

?

ell Ruijsenaars

trig Ruijsenaars-
Macdonald

trig (momenta)
affine Hecke alg

non-r/t

rational (momenta)
degenerate AHA

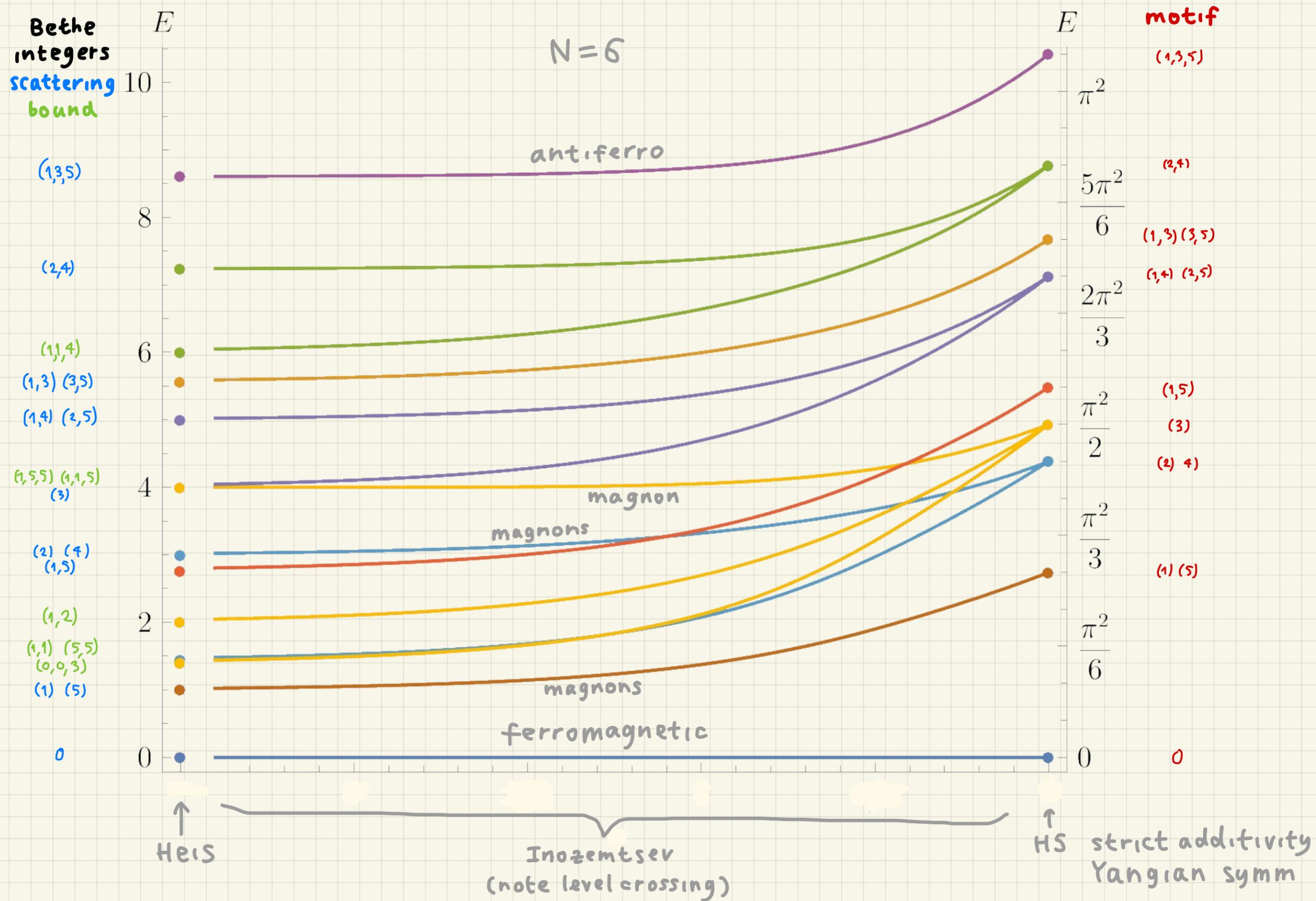
~ Lieb-Liniger?

ell Cal-Sut

trig Cal-Sut

towards grand unified theory for
quantum-integrable long-range spin chains?

Isotropic level example



Isotropic level exact spectra

Heis XXX '28

method

(e.g coord) Bethe ansatz

ansatz for $\Psi(n_1, \dots, n_M)$
coord of ↓'s

$$\sum_{\pi \in S_M} A_{p,\pi} e^{i p \cdot n_\pi}$$

↑ plane waves
↑ account for (contact) interactions

Schrödinger eqn fixes

$A_{p,\pi}$ as fn of p
(up to normalisation)

periodicity fixes

values of p_1, \dots, p_M
Bethe ansatz eqs

$$p_m = \frac{2\pi}{N} I_m + \sum_{m' (\neq m)} \Theta_H(p_m, p_{m'})$$

dispersion

$$\epsilon_H(p) = 4 \sin^2 \frac{p}{2}$$

energy

$$E_H = \sum_{m=1}^M \epsilon_H(p_m)$$

functionally additive

Inozemtsev '90

$$\begin{array}{ccc} \delta & \xrightarrow{\kappa \rightarrow 0} & \frac{1}{\sin^2} \\ \downarrow & & \\ N \rightarrow \infty & & \\ \frac{1}{\sinh^2} & & \end{array}$$

extended CBA
→ conn to ell Cal-Sut

$$\sum_{\pi \in S_M} \tilde{\Psi}_{\tilde{p}}(n_\pi) e^{i(-\tilde{p} + p) \cdot n_\pi}$$

↑ ~params φ_m, \tilde{p}_m
doubly quasiperiodic
simple poles at equal args

if identify $\tilde{p}_m = \lambda(p_m)$, rapidity function

$\tilde{\Psi}_{\tilde{p}}(x_1, \dots, x_M)$ must be eigenfn of

$$\tilde{H}_{\text{eCS}}^* \sim -\frac{1}{2} \sum_{m=1}^M \partial_{x_m}^2 + 2 \sum_{m < m'} \delta(x_m - x_{m'}) \rightarrow$$

which are known (complicated)

values of p_1, \dots, p_M
Bethe ansatz eqs

$$p_m = \frac{2\pi}{N} I_m + \frac{1}{N} \varphi_m \quad \begin{array}{l} 1 \leq I_m \leq N-1 \\ I_{m+1} \geq I_m \end{array}$$

$$\epsilon_I(p) \sim \bar{\theta}(p) - (\bar{\xi}(p) + \text{cst } p)^2 \rightarrow$$

$$E_I = \sum_{m=1}^M \epsilon_I(p_m) - \tilde{U}$$

'quasi-additive'

$$\tilde{U} = -\frac{1}{2} \sum \tilde{p}_m^2$$

Haldane '88 - Shastry '88

connection to trig Cal-Sut

$$\tilde{\Psi}(\omega^{n_1}, \dots, \omega^{n_M})$$

↑ symmetric polynomial $\omega \equiv e^{2\pi i/N}$
deg < N in each argument
= 0 at equal arguments

$\tilde{\Psi}(e^{ix_1}, \dots, e^{ix_M})$ must be eigenfn of

$$\tilde{H}_{\text{tCS}}^* = -\frac{1}{2} \sum_{m=1}^M \partial_{x_m}^2 + 2 \sum_{m < m'} \frac{(n/N)^2}{\sin(\frac{\pi}{N}(x_m - x_{m'}))^2}$$

which are known (simple ~Jack)

nothing periodicity is built in

$$p_m = \frac{2\pi}{N} \mu_m \quad \text{motif } \begin{array}{l} 1 \leq \mu_m \leq N-1 \\ \mu_{m+1} > \mu_m + 1 \end{array}$$

$$\epsilon_{HS}(p) = \frac{1}{2} p(2\pi - p) \rightarrow$$

$$E_{HS} = \sum_{m=1}^M \epsilon_{HS}(p_m)$$

strictly additive