A Lazy Logician's Guide to Linear Logic

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# A Lazy Logician’s Guide to Linear Logic

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O. Introduction

The purpose of these notes is to provide an accessible introduction to the main concepts of linear logic. The seminal work in this new area of research is the paper by Jean-Yves Girard, "Linear Logic", Theoretical Computer Science, 50 (1987), 1-102. (subsequently, Girard [1987]). This lengthy report is not recommended for light bedtime reading; its quite substantial content demands many hours in a quiet room and a good supply of pencils and paper. Over the last few years, the literature on linear logic has been growing at a rather more than linear rate. However, the bulk of these papers, including introductory expositions such as Lafont [1988a] and Seely [1989], are to be found in journals or conference proceedings whose intended audience consists of computer scientists and/or category theorists. We hope that these notes will be of use to logicians of other persuasions. We will attempt to make perspicuous the rather elegant mathematics underlying linear logic.

Our basic reference is Girard [1987]. We work through much of that paper, filling out details, supplying proofs of lemmas, rearranging things etc. Section 5 also incorporates work from Girard, Lafont and Taylor [1989]. Departures from and modifications to the original are noted, as are references to other papers. One departure is our omission of propositional quantifiers, specifically the treatment of them in Chapters 3 and 4, on coherent spaces and proof nets respectively, of Girard [1987]. Readers keen to go higher order will need a good grip on ground level stuff; hopefully they'll get that here. In spite of justifiable complaints by a number of researchers (e.g. Avron [1988], Seely [1989]) about Girard's choice of notation for his logical operators,
we have stuck to the original. Appendix A consists of a translation table which may be helpful to readers of Seely [1989], Avron [1988] or anyone familiar with relevance logics.

To get a feel for what linear logic is on about, consider Hilbert-style axiomatic systems where the only rule of inference is modus ponens (mp), as in Avron [1988]. We restrict our attention to the implicational fragment of a language for propositional logic. We can characterize a linear proof as one in which every occurrence of a formula other than the last is used exactly once as a premise of MP. In contrast, the proofs appropriate to the system $R \rightarrow$ (the implicational subsystem of the relevance logic $R$) are those in which every occurrence of a formula other than the last is used at least once as a premise of MP. In terms of Gentzen sequent calculi, systems for linear logic have neither contraction nor thinning among their structural rules; in the system for $R$, thinning is dropped but contraction is retained. Thinning (also known as weakening) gives rise to spurious dependencies ('irrelevancies') such as $A \rightarrow (B \rightarrow A)$; in a proof of such a formula, the hypothesis $B$ is not used, so we have wastage. Contraction allows an occurrence of a formula to be used over and over again; roughly, this means that the first time you get something, you pay for it, but after that you're getting stuff for free. In linear logic, nothing is wasted, and nothing is gotten for free. (For more on the intuitive content of linear logic, see Section 4 below.)

Linear logic is attracting quite a lot of attention in the computer science community. Regrettably, the author is almost totally ignorant of computer science so is in no
position to say anything about the significance of linear logic for this field. The reader is referred to Lafont [1988b], which is an extended version of Girard and Lafont [1987]. Lafont introduces the 'Linear Abstract Machine' as a technique of implementing functional programming languages based on intuitionistic linear logic, i.e. in the sequent calculus for this logic, the right sides of sequents contain at most one formula; see Appendix B for the two-sided sequent calculi for 'classical' and intuitionistic linear logic. The memory allocation in Lafont's implementation is such that there is no need for a garbage collector; very roughly, the sort of waste produced by thinning is what you would need a garbage collector for. In addition, the Linear Abstract Machine permits a synthesis of strict and lazy evaluation strategies: 'call by value' along side 'call by need'! (For those who have read ahead, the multiplicative $A \otimes B$ is a type of strict pairs while the additive $A + B$ is a type of lazy pairs; we have no need to say any more about the matter here.) For more on the import of linear logic for the study of parallelism, see the introduction to Girard [1987] and the essay "Towards a Geometry of Interaction", Girard [1989]. The latter presents 'a program for proof theory inspired by its growing connections with computer science'.

These notes are based on material presented in a series of fifteen seminars held at the University of Melbourne, March–July 1991. The author wishes to thank, for invaluable comments and corrections, the regular participants: Jacinta Cavington, Allen Hazen, David Odell (University of Melbourne) and Daniel Mahler (Monash University), and the irregular participants: Kevin Davey, John Collins, Lloyd Humberstone (Monash), David Kinnick and Harald Søndergaard (Melbourne).
1. Sequent calculi

1.0 Additives and multiplicatives

Suppose our lazy logician has a dim recollection of reading Schwichtenberg [1971], remembering only that a one-sided sequent calculus for classical logic means that there is only half as many rules to write out and deal with. Each two-sided sequent of the form

\[ A_1, \ldots, A_n \vdash B_1, \ldots, B_m \]

is replaced by:

\[ \vdash \neg A_1, \ldots, \neg A_n, B_1, \ldots, B_m. \]

Being a bit woolly on the details, our lazy logician hesitates between the following two possible rules for conjunction:

\[ \begin{array}{c}
\vdash A, \Delta \\
\vdash B, \Delta \\
\hline
\vdash A \land B, \Delta
\end{array} \quad \begin{array}{c}
\vdash A, \Delta \\
\vdash B, \Gamma \\
\hline
\vdash A \land B, \Delta, \Gamma
\end{array} \]

If we have the rules of thinning and contraction:

\[ \begin{array}{c}
\vdash \Delta \\
\hline
\vdash A, \Delta
\end{array} \quad \text{and} \quad \begin{array}{c}
\vdash A, A, \Delta \\
\hline
\vdash A, \Delta
\end{array} \]

respectively, then with thinning we can derive the second of the conjunction rules from the first, and with contraction we can derive the first from the second.

Our lazy logician agrees to drop thinning and contraction: at the least, there will be a few less cases to worry about in
the proof of Gentzen’s Hauptsatz (Cut Elimination theorem). Moreover, contraction is responsible for the undecidability of predicate calculus. The sub-formula property (corollary of the Hauptsatz) yields a (theoretical) decision procedure for predicate calculus, provided a bound can be placed on the length of any sequent occurring in a cut-free proof. In the absence of contraction, all sequents occurring in a cut-free proof have length less than or equal to the length of the end sequent. (This is noted in Girard [1989], p.79, and elsewhere.)

In linear logic we have two quite different conjunctions:

- an **additive** conjunction \& (called ‘with’), characterized by the rule

\[
\frac{\Gamma, A, \Delta \quad \Gamma, B, \Delta}{\Gamma, A \& B, \Delta} \quad (\&)
\]

i.e. two copies of the same parametric sequence are identified, giving a type of implicit contraction; and

- a **multiplicative** conjunction \otimes (called ‘times’), characterized by the rule

\[
\frac{\Gamma, A, \Delta \quad \Gamma, B, \Pi}{\Gamma, A \otimes B, \Delta, \Pi} \quad (\otimes)
\]

i.e. parametric sequences are accumulated.

(The terms ‘additive’ and ‘multiplicative’, as well as the buzz word ‘linear’, have their origins in the coherent semantics for linear logic: see Section 5 below. With respect to the relevance logic tradition, ‘additive’ corresponds to ‘extensional’ and ‘multiplicative’
to 'intensional'.)

The absence of thinning and contraction also gives rise to two distinct verum-like constants:

- an additive one, $T$, characterized by the axiom scheme:
  $$\vdash T, \Delta$$
  where $\Delta$ is any sequence of formulae; and

- a multiplicative one, $1$, characterized by the axiom:
  $$\vdash 1$$

A surprising feature of linear logic is that although it is constructive — propositional intuitionistic logic is faithfully translatable into modal linear logic — it retains an involutive negation, $\neg \neg$. (We pronounce it 'perp'.) As a consequence, de Morgan-style duality is rife within linear logic:

- the additive disjunction $\oplus$ (‘plus’) is the de Morgan dual of $\&$;
- the multiplicative disjunction $\otimes$ (‘par’) is the de Morgan dual of $\otimes$;
- the additive falsum-like constant $\top$ is the linear negation of $T$; and
- the multiplicative falsum-like constant $\bot$ is the linear negation of $1$.

(We pronounce $\bot$ as 'eet'.)
We get a rather sleek syntax and sequent calculus for linear logic if we take the symbols $\Theta, \Theta, \Theta$ and $\bot$ as primitive, in addition to $\&$, $\otimes$, $\top$ and $\bot$, and then introduce $(\neg)\text{'}$ by definition. A symbol for linear implication, $\rightarrow$, (‘entails’; we usually call it ‘lollipop’) will also be introduced by definition:

$$A \rightarrow B \iff A \otimes B$$

i.e. the multiplicative analogue of material implication.

Justification for this act of definitional fiat with respect to linear negation and linear implication will be found in the phase space semantics (Section 2).

Historically, Girard first worked with linear implication, by way of a ‘decomposition’ of intuitionistic implication (see sections 3-3 and 5-3 below) and then built the rest of the logic around it.

Some readers may be justifiably perplexed by the weird symmetries in Girard’s notation:

<table>
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<th>CONJUNCTION</th>
<th>VERUM-LIKE</th>
<th>DISJUNCTION</th>
<th>FALSUM-LIKE</th>
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<td>ADDITIVE</td>
<td>$&amp;$</td>
<td>$\top$</td>
<td>$\oplus$</td>
</tr>
<tr>
<td>MULTIPLICATIVE</td>
<td>$\otimes$</td>
<td>$\bot$</td>
<td>$\circ$</td>
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In particular, why doesn’t the notation reflect the de Morgan dualities? Girard seems to want to stress that the multiplicative connectives distribute over their notationally similar additives (see section 1-3 below). One may also derive some consolation from the fact that it will turn out that the category of finite dimensional vector spaces over a field $k$, $\text{Vect}_k$, whose morphisms are linear maps,
is a model of linear logic. The multiplicative \( \otimes \) is interpreted as the tensor product \( \otimes \), the additive \( + \) is the direct sum \( \oplus \) and linear negation \( (-)^\perp \) is the duality operator \( (-)^* \). (See section 5.4 below.) So linear logic has something to do with linear algebra.

1.1 The sequent calculi LL and LLM.

In later work we will need to refer to the pure multiplicative fragment, i.e. the symbols \( \otimes \) and \( \& \) but not the constants \( 1 \) and \( \perp \), so we allow for this in our definitions here.

**Definition 1.1.0:**

We have symbols of the following kinds:

(i) connective symbols: \( \otimes, \& , \oplus, \oplus \);

(ii) constants: \( 1, \perp, T, O \);

(iii) literals:

(a) propositional letters: \( P, Q, R, \ldots \)

(b) the duals of propositional letters: \( P^\perp, Q^\perp, R^\perp, \ldots \)

The language \( L \) of (propositional) linear logic is defined as follows:

- constants and literals are formulae in \( L \);
- if \( A \) and \( B \) are formulae in \( L \) then so are \( A \otimes B, A \& B, A \oplus B \) and \( A \oplus B \).

The language \( L_m \) of the pure multiplicative fragment of linear logic is defined as follows:

- literals are formulae in \( L_m \);
- if \( A \) and \( B \) are formulae in \( L_m \) then so is \( A \otimes B \) and \( A \& B \).
Constants and literals are atomic formulae of $L$; literals are atomic formulae of $L_m$.

**Definition 1.1.1:**
The expression $(A)^{\perp}$ where $A$ is a formulae in $L$ ($L_m$) is defined as follows:

$$(1)^{\perp} \overset{df}{=} 1$$

$$(0)^{\perp} \overset{df}{=} 0$$

$$(T)^{\perp} \overset{df}{=} T$$

$$(F)^{\perp} \overset{df}{=} F$$

if $p$ is a propositional letter, $(p^{\perp})^{\perp} \overset{df}{=} p$;

if $A$ and $B$ are formulae in $L$ ($L_m$),

$$(A \otimes B)^{\perp} \overset{df}{=} (A)^{\perp} \otimes (B)^{\perp}$$

$$(A \& B)^{\perp} \overset{df}{=} (A)^{\perp} \& (B)^{\perp}$$

$$(A \rightarrow B)^{\perp} \overset{df}{=} (A)^{\perp} \& (B)^{\perp}.$$  

The expression $A \rightarrow B$, where $A$ and $B$ are formulae in $L$, is defined as follows:

$$A \rightarrow B \overset{df}{=} (A)^{\perp} \& B.$$  

Parentheses may be omitted where appropriate.

Note that formulae in $L$ are automatically in negation-normal form. Capital Roman letters $A, B, C, D, E$ will be used solely as metalinguistic variables ranging over $L$ (or $L_m$). Metalinguistically, we have the identity $A^{\perp \perp} = A$.  

Definition 1.1.2:

An \( \mathcal{L} \)-sequent (\( \mathcal{L}_m \)-sequent) is an expression of the form

\[ \vdash A_1, \ldots, A_n \]

where \( A_1, \ldots, A_n \) are formulae in \( \mathcal{L} \) (\( \mathcal{L}_m \)).

We use capital Greek letters \( \Delta, \Gamma, \Sigma, \Theta \) to denote sequences of formulae.

If \( \Delta \) is the empty sequence, then \( \vdash \Delta \) has the same meaning as \( \vdash \bot \).

The sequent calculus \( \mathcal{LL} \) consists of the axiom schemes:

- **Identity**: \( \vdash A, A^\Delta \) where \( A \) is a propositional letter.
- **Additive**: \( \vdash \top, \Delta \)
- **Multiplicative**: \( \vdash \bot \)

and the following inference rules:

- **Structural**: \( \vdash \Delta \quad (\text{EXCH}) \quad \vdash \Delta, \Delta, \Gamma \quad (\text{CUT}) \)

  where \( \sigma \) is any permutation of \( \Delta \)

- **Additive**: \( \vdash A, \Delta \quad \vdash B, \Delta \quad (\&) \quad \vdash A, \Delta \quad (\text{FST}) \quad \vdash B, \Delta \quad (\text{snd}) \)

  \( \vdash A \otimes B, \Delta \)

- **Multiplicative**: \( \vdash A, \Delta \quad \vdash B, \Gamma \quad (\&\& \quad \vdash A, \Delta \quad (\otimes) \quad \vdash A \otimes B, \Delta \quad (\&\& \quad \vdash A \otimes B, \Delta \quad (\&\& \quad \vdash \Delta \quad (\bot) \quad \vdash \bot, \Delta \)

The sequent calculus \( \mathcal{LLM} \) consists of the axiom scheme of identity and the inference rules (\( \text{EXCH} \), (\( \text{CUT} \)), (\( \otimes \)) and (\( \&\& \)).
The restriction of the identity axiom scheme to propositional
letters is due to the following:

\[
\begin{align*}
\text{(i)} & \quad \frac{\vdash A, A^\perp}{\vdash A^\perp, A} \quad \frac{\vdash B, B^\perp}{\vdash B^\perp, B} \\
& \quad \frac{\vdash A^\perp \& B^\perp, A}{\vdash A^\perp, A^\perp \& B^\perp} \quad \frac{\vdash A^\perp \& B^\perp, B}{\vdash B, A^\perp \& B^\perp} \\
& \quad \frac{\vdash A^\perp \& B^\perp, A^\perp \& B^\perp}{\vdash A \& B, A^\perp \& B^\perp} \\
\end{align*}
\]

\text{(ii)} \quad \vdash T, 0

\text{(iii)} \quad \frac{\vdash A, A^\perp}{\vdash B, B^\perp} \quad (\otimes)
\quad \frac{\vdash A \otimes B, A^\perp, B^\perp}{\vdash A \& B, A^\& B^\perp} \quad (\otimes)

\text{(iv)} \quad \frac{\vdash 1}{\vdash 1, 1} \quad (1)

In view of the inference rule \((\otimes)\), an \(L\)-sequent or
\(L_m\)-sequent

\[\vdash A_1, \ldots, A_n\]

will be given the same semantic interpretation as the formula

\[A_1 \& \cdots \& A_n\]

(Associativity is not going to be a problem).

In virtue of the structural rule \((\text{EXCH})\), we could work
with multi-sets of formulae rather than sequences of formulae;
we still have to distinguish occurrences of formulae.
Gentzen's proof of the Hauptsatz for the classical sequent calculus \( \text{LK} \) can be readily modified (working on cuts directly rather than on mixes, the latter requiring thinning) to give

**Proposition 1.1.3:** (Hauptsatz for \( \text{LL} \))

If \( \Pi \) is a proof of \( \Gamma \Delta \) in \( \text{LL} \)

then \( \Pi \) can be transformed into a proof \( \Pi' \) of \( \Gamma \Delta \)

in which the inference rule (cut) is not used.

Thistlewaite, McRobbie and Meyer [1988] have independently devised a one-sided sequent calculus strikingly similar to \( \text{LL} \). Their system, for the relevance logic \( \text{R} \) minus distribution, is the same as \( \text{LL} \) plus contraction. 'Distribution' in this context is an axiom scheme expressing the distributivity of additive conjunction over additive disjunction; it is conventionally adjoined to the relevance logic \( \text{R} \). In the notation of linear logic, the axiom scheme is

\[
A \& (B \oplus C) \rightarrow (A \& B) \oplus C.
\]

With contraction, one readily derives

\[
A \& (B \oplus C) \rightarrow (A \& B) \oplus (A \& C).
\]

For more on the relation between linear logic and relevance logics, see Avron [1988].

In Appendix B, we set out the two-sided sequent calculus for standard, 'classical', linear logic and for intuitionistic linear logic.
1.2 Some theorems of LL

Definition 1.2.0:
A is a theorem of LL iff the sequent \( \vdash A \) is provable in LL.
A and B are provably equivalent in LL, written \( A \equiv B \),
iff both \( A \Rightarrow B \) and \( B \Rightarrow A \) are theorems of LL.

Lemma 1.2.1:
The following are provable equivalences in LL (omitting the subscript LL):

(i) associativity and commutativity:
\[
\begin{align*}
A \& (B \& C) & \equiv (A \& B) \& C & \quad & A \& B & \equiv B \& A \\
A \oplus (B \oplus C) & \equiv (A \oplus B) \oplus C & \quad & A \oplus B & \equiv B \oplus A \\
A \otimes (B \otimes C) & \equiv (A \otimes B) \otimes C & \quad & A \otimes B & \equiv B \otimes A \\
A \# (B \# C) & \equiv (A \# B) \# C & \quad & A \# B & \equiv B \# A
\end{align*}
\]

(ii) constants:
\[
\begin{align*}
T \& A & \equiv A & \quad & 0 \oplus A & \equiv A \\
1 \oplus A & \equiv A & \quad & \perp \# A & \equiv A \\
0 \otimes A & \equiv 0 & \quad & T \# A & \equiv T \\
1 \Rightarrow A & \equiv A & \quad & A \Rightarrow \perp & \equiv A^\perp \\
0 \Rightarrow A & \equiv T & \quad & A \Rightarrow T & \equiv T
\end{align*}
\]

(iii) \[
\begin{align*}
A \Rightarrow B & \equiv B^\perp \Rightarrow A^\perp \\
(A \otimes B) \Rightarrow C & \equiv A \Rightarrow (B \Rightarrow C) \\
A \Rightarrow (B \# C) & \equiv (A \Rightarrow B) \# C
\end{align*}
\]
(iv) distributivity of multiplicatives over 'opposite' additives:

\[
A \otimes (B \circ C) \equiv (A \otimes B) \circ (A \otimes C)
\]
\[
A \& (B \& C) \equiv (A \& B) \& (A \& C)
\]
\[
(A \oplus B) \setminus C \equiv (A \setminus C) \& (B \setminus C)
\]
\[
A \setminus (B \& C) \equiv (A \setminus B) \& (A \setminus C)
\]

The following are theorem schemes for LL:

(vi) half-distributivity of multiplicatives over 'like' additives:

\[
A \otimes (B \& C) \setminus (A \otimes B) \& (A \otimes C)
\]
\[
(A \& B) \circ (A \& C) \setminus A \& (B \& C)
\]
\[
(A \setminus C) \& (B \setminus C) \setminus ((A \& B) \setminus C)
\]
\[
(A \setminus B) \circ (A \setminus C) \setminus (A \setminus (B \& C))
\]

(vii) half-distributivity of additives:

\[
(A \& B) \oplus (A \& C) \setminus A \& (B \& C)
\]
\[
A \oplus (B \& C) \setminus (A \oplus B) \& (A \oplus C)
\]

(viii) weak distributivity of multiplicatives:

\[
A \otimes (B \& C) \setminus (A \otimes B) \& C
\]

Proof: We will only exhibit proofs for (vii) and (viii). The reader is invited to try one of the others for their own edification. The first equivalence in (iv) is 'borrowed' from linear algebra: the distributivity of the tensor product over direct sum. In (vii), read \& as 'meet', \oplus as 'join' and \setminus as inclusion; such a translation yields general properties of lattices (see section 2.1 below).
We omit exchanges.

(vii)

\[ \frac{\vdash A, A^\perp}{\vdash A\otimes B^\perp} \quad \frac{\vdash B, B^\perp}{\vdash B, A^\perp\otimes B^\perp} \quad \frac{\vdash A, A^\perp}{\vdash B\otimes C, A^\perp\otimes B^\perp} \quad \frac{\vdash C, C^\perp}{\vdash C, A^\perp\otimes C^\perp} \]

\[ \vdash A\otimes B^\perp, A\otimes (B\otimes C) \]

\[ \vdash A^\perp\otimes C^\perp, A\otimes (B\otimes C) \]

\[ \vdash (A^\perp\otimes B^\perp) \otimes (A^\perp\otimes C^\perp), A\otimes (B\otimes C) \]

\[ \vdash ((A^\perp\otimes B^\perp) \otimes (A^\perp\otimes C^\perp)) \otimes (A\otimes (B\otimes C)) \]

(viii)

\[ \frac{\vdash B^\perp, B}{\vdash C^\perp, C} \quad \frac{\vdash A, A^\perp}{\vdash B, B^\perp\otimes C^\perp, C} \]

\[ \vdash A\otimes B, A^\perp, B^\perp\otimes C^\perp, C \]

\[ \vdash A^\perp \otimes (B^\perp\otimes C^\perp), A\otimes B, C \]

\[ \vdash A^\perp \otimes (B^\perp\otimes C^\perp), (A\otimes B) \otimes C \]

\[ \vdash (A^\perp \otimes (B^\perp\otimes C^\perp)) \otimes ((A\otimes B) \otimes C) \]

The restricted sequent calculus LLM is highly conservative.

Observation 1.2.2:

Let $\Pi$ be a cut-free proof of $\vdash \Delta$ in LLM, and let $\vdash A_i,A_i^\perp,\ldots,\vdash A_n,A_n^\perp$ be a list of all the instances of the identity axiom scheme which occur in $\Pi$. Then the atomic subformulae (literals) of $\Delta$ are exactly $A_1,A_1^\perp,\ldots,A_n,A_n^\perp$.

This is established by a simple induction on the proof of $\vdash \Delta$. 
Observation 1.2.3:
Let \( \Gamma \Delta \) be any provable sequent of LL\(M \), and let \( A_1, \ldots, A_n \) be a list of all occurrences (as atomic Subformulae) of positive literals in \( \Delta \). Then the negative literals occurring in \( \Delta \) are exactly \( A_1^+, \ldots, A_n^+ \).

i.e. cuts destroy literals in dual pairs.

Observation 1.2.4:
(i) A formula of \( L_m \) of the form
\[
A \rightarrow (A \otimes A)
\]
is a theorem of LL\(M \) iff \( A \) is a theorem of LL\(M \).

(ii) No formula of \( L_m \) of the form
\[
(A \otimes B) \rightarrow A
\]
where \( B \neq A \otimes A^+ \) and \( B \neq A^+ \otimes A \), is a theorem of LL\(M \).

Formulae of type (i) are characteristic of contraction. Those of type (ii) are symptoms of thinning; \( (A \otimes B) \rightarrow A \) is provably equivalent to \( A \rightarrow (B \rightarrow A) \) by the associativity of \( \otimes \).

When we extend the sequent calculus from LL\(M \) to LL, we lose the balance of positive and negative literals (and hence have a more interesting logic).

However, the analogue for \( L \) of 1.2.4(i) still holds for LL and formulae of \( L \) of the form \( (A \otimes B) \rightarrow A \) are not, in general, theorems of LL (but there are more cases to exclude).
Let us consider the axioms schemes and inference rules of **LL** a little more closely. The axiom scheme for **T** is generous, but harmlessly so. Observe that if an instance of the axiom scheme for **T** occurs in an **LL** proof, then there must be an occurrence of **T** in the end sequent (so **T**'s can't be 'cut-out' completely). The rule (⊥) allows us to 'thin-in' as many copies of ⊥ as we like. The rules (Fst⊥) and (Snd⊥) also involve a type of thinning, at the level of subformulae, but are still conservative with respect to parametric sequences. Otherwise put, these rules, like (⊗) and (⊕), act locally, i.e. only on principal formulae. In contrast, the rule (⊗), with its requirement that the two premiss sequents contain the same parametric sequences, involves a global constraint. In our later work with an alternative proof procedure (the system of 'proof nets', in section 6) we will have trouble dealing with ⊤ because of this constraint. (The problem is familiar to relevance logicians: see Dunn [1986] or Avron [1988].)
2. Phase space semantics

In this section we give an algebraic semantics for linear logic and prove the soundness and completeness of LL with respect to this semantics. This is, as Girard puts it, a 'Tarskian-style' semantics and in virtue of completeness we get an 'official blessing' for the system LL. In Section 5 we present an alternative, 'disturbing' semantics in terms of coherent spaces, which Girard takes to be more in the tradition of Heyting. Coherent spaces emerge from a reworking of Dana Scott's theory of domains (models for intuitionistic logic / typed λ-calculus). Girard uses them to show how linear implication results from a 'decomposition' of intuitionistic implication. In the framework of coherent spaces, we can naturally associate a semantic object with each proof in LL.

But first we must bestow upon LL a nihil obstat.

2.0  Phase spaces and facts

**Definition 2.0.0:**
A phase space is a quadruple \( \langle P, \cdot, I, \bot_P \rangle \) consisting of
(i) a commutative monoid \( \langle P, \cdot, I \rangle \),
    i.e. satisfying (a) for all \( p, q, r \in P \), \( p(qr) = (pq)r \)
    (b) for all \( p \in P \), \( p \cdot I = 1 \cdot p = p \)
    and (c) for all \( p, q \in P \), \( pq = qp \)
    and
(ii) a distinguished subset \( \bot_P \subseteq P \).
The elements \( p \in P \) are called phases, and \( \bot_p \) is the set of orthogonal phases of \( \langle P, \cdot, \bot, \bot_p \rangle \).

As the first of many abuses of notation, we shall refer to \( P \) as a phase space and drop the subscript \( \bot_p \) when the context makes it clear. We are going to interpret formulae in \( L \) by certain subsets of \( P \); not surprisingly, \( \bot \) will be the interpretation of the constant \( \bot \). In what follows, we will re-use (and abuse) the connective symbols of \( L \) in giving definitions of the corresponding semantic operations.

**Definition 2.0.1:**

Let \( G \subseteq P \) be any subset. We define the dual of \( G \), written \( G^\perp \), as follows:

\[
G^\perp \overset{\text{def}}{=} \{ q \in P \mid \text{for all } p \in G (pq \leq \bot)^\frac{1}{2} \}
\]

So \( G^\perp \) consists of those \( q \) which send all of \( G \) into \( \bot \). Observe that for any \( p \in P \),

\[
\{p^\perp\} = \{ q \in P \mid pq \leq \bot^\perp \}
\]

and for any \( G \subseteq P \),

\[
G^\perp = \bigcap \{ p^\perp \mid p \in G \}
\]

**Definition 2.0.2:**

Let \( G \subseteq P \). \( G \) is a fact of \( P \) if \( G^{\perp \perp} = G \).
A fact \( G \) is verified in \( P \) if \( 1 \in G \).
Girard thinks of the $p \in G$, where $G$ is a fact of $P$, as phases 'between the fact and its verification', or tasks to be undertaken in order to verify $G$. When $1 \in G$, there are no tasks to do; $G$ is verified in $P$. If we write ' $p \models G$ ' instead of ' $p \in G$ ' (as Girard does in his introductory discussion) the connection with Kripke-style semantics becomes clearer. (cf. Dunn[1986] on the semantics of relevance logics.)

**Lemma 2.0.3:** (Girard [1987])

Let $G, H \in P$.

(i) \( G \subseteq G^{++} \);

(ii) \( G^{++} = G^+ \);

(iii) if $G \subseteq H$ then $H^+ \subseteq G^+$;

(iv) $G$ is a fact of $P$ iff $G = H^+$ for some $H \in P$.

(v) if $H$ is a fact of $P$ and $G \subseteq H$ then $G^{++} \subseteq H$.

**Proof:**

(i) Let $p \in G$. Then for all $q \in G^+$, $pq \in \perp$. But

\[ G^{++} = \{ p \in P \mid \text{for all } q \in G^+ (pq \in \perp) \}. \]

Hence $p \in G^{++}$.

(ii) By (i), $G^+ \subseteq G^{+++}$.

Let $q \in G^{+++}$. Then for all $p \in G^{++}$, $pq \in \perp$. Since $G \subseteq G^{++}$, we have that for all $p \in G$, $pq \in \perp$.

Hence $q \in G^+$.

(iii) Similar argument to (ii).

(iv) Immediate consequence of (iii).
(v) Let $S = \bigcap \{ H \subseteq P \mid G \subseteq H \text{ and } H = H^{-+} \}$. Then $G \subseteq S \subseteq G^{++}$. By (iii), $G^{++} \subseteq S^{++} \subseteq G^{+++}$. By (ii), $G^{+++} = G^{++}$, hence $G^{++} = S^{++}$.

To obtain $S = S^{++}$, we need the following result.

**Lemma 2.0.4:** (Girard [1987])

If $\{G_i\}_{i \in I}$ is any family of facts of $P$ then

$$\bigcap_{i \in I} G_i = (\bigcup_{i \in I} G_i^+)^-.$$

Hence the intersection of any family of facts is a fact.

**Proof:**

First observe that

$$\bigcup_{i \in I} G_i^+ = \bigcup_{i \in I} \{ q \mid pq \in L \}.$$

Let $q \in \bigcup_{i \in I} G_i$. Then for some $i \in I$, and for all $p \in G_i$, $pq \in L$. Thus if $p \in \bigcap_{i \in I} G_i$ then $pq \in L$, hence $p \in (\bigcup_{i \in I} G_i^+)^-$. Conversely, let $p \in (\bigcup_{i \in I} G_i^+)^-$. Then for each $i \in I$, if $q \in G_i^+$ then $q \in \bigcup_{i \in I} G_i^+$ hence $pq \in L$. Hence $p \in \bigcap_{i \in I} G_i^{++}$. Since $G_i = G_i^{++}$, we have $p \in \bigcap_{i \in I} G_i$.

**Corollary 2.0.5:**

For any $G \subseteq P$,

$$\left( \bigcup_{i \in I} G_i^{++} \mid p \in G_i \right)^{++} = G^{++}.$$
Definition 2.0.6:
\[ \text{Fact}(P) = \left\{ G \subseteq P \mid G = G^{++} \right\} \]

Fact(P) is partially ordered by inclusion. Note that \( G \in \text{Fact}(P) \)
iff \( G \) is the intersection of some family of duals of singletons,
\[ G = \cap \{ \xi P^+ \mid p \in H \} \]
for some set \( H \subseteq P \).

Lemma 2.0.3 (ii)-(iii) implies that \( (-)^{++} \) is a closure
operator on \( P \). So by detouring via a standard result
in lattice theory (e.g., Burris and Sankappanavar [1981], I.5.2)
we could have bypassed 2.0.4 and concluded that
the poset \( \langle \text{Fact}(P), \subseteq \rangle \) is a complete lattice,
with
\[ \bigwedge_{i \in \mathbb{I}} G_i = \cap_{i \in \mathbb{I}} G_i \quad \text{and} \quad \bigvee_{i \in \mathbb{I}} G_i = \left( \bigcup_{i \in \mathbb{I}} G_i \right)^{++} \]
where \( \{ G_i \}_{i \in \mathbb{I}} \) is any family of facts of \( P \).

Examples 2.0.7:
(i) \( \bot \in \text{Fact}(P) \) since \( \bot = \{ \bot^+ \} \).
(ii) Define \( 1 \uparrow \bot \) so \( 1 \in \text{Fact}(P) \). Now
\[ 1 = \{ p \in P \mid \text{for all } q \in \bot (pq \in \bot) \} \]
Thus \( 1 \in 1 \) and if \( p \in 1 \) and \( q \in 1 \) then \( pq \in 1 \). Hence
the \( 1 \) is a submonoid of \( P \). Moreover, if \( p \in 1 \) and \( q \in G \),
for any subset \( G \subseteq P \), then \( pq \in G^{++} \).

Observe that \( G \in \text{Fact}(P) \) is verified in \( P \) exactly when \( G \)
contains \( 1 \):
\[ 1 \in G \iff G^\perp \subseteq \bot \iff 1 \subseteq G \]
assuming \( G = G^{++} \).
(iii) Define $T = \emptyset^\perp$, hence $T = P \in \text{Fact}(P)$.
(iv) Define $0 \triangleq T^\perp$. Then $0 \in \text{Fact}(P)$, and since

$$0 = \{ p \in P \mid \forall q \in P \ (pq \in \bot) \}$$

we have $0 \leq G$ for all $G \in \text{Fact}(P)$.
Trivially, $G \leq T$ for all $G \in \text{Fact}(P)$,
hence $< \text{Fact}(P), \leq >$ is a bounded lattice. (Henceforth, we'll use $'\text{Fact}(P)'$ to refer to both the lattice and its underlying set.)

The general picture of $\text{Fact}(P)$ is as follows:

By construction, $(-)^\perp$ is an involution on $\text{Fact}(P)$, but it is well removed from classical negation. We certainly don't have, for all $G \in \text{Fact}(P)$, either $1 \in G$ or $1 \in G^\perp$. Moreover, there is nothing to rule out the possibility of both $1 \in G$ and $1 \in G^\perp$, for some $G \in \text{Fact}(P)$. However, we do have the following

**Lemma 2.0.8:**
There is a $G \in \text{Fact}(P)$ such that $1 \in G$ and $1 \in G^\perp$ iff $1 \in \bot_P$. 
Proof:
Suppose $G \in \text{Fact}(P)$ is such that $1 \leq G$ and $1 \leq G^\perp$. Then
$G^\perp \leq 1$ and $1 \leq G^\perp$. Hence $1 \leq 1$, so $1 = 1$.

Conversely, suppose $1 = 1$. But $1 = 1$ always, so we are done.

Phase spaces $\langle P, \cdot, 1, 1_p \rangle$ such that $1 \leq 1_p$ are a bit weird, if not outright pathological. Now as a
corollary to the Hauptsatz, we have that $1 \leq 1$ is
not provable in $LL$. (Note that this only gives a weak
consistency; the provability of the empty sequent is only
really dangerous in the presence of weakening.) Once we
have proved the completeness of $LL$, we'll have a
guarantee that not all phase spaces are pathological.

2.1 The Additives

Obviously, the additives are the lattice operations on $\text{Fact}(P)$.

Definition 2.1.0:
Let $G, H \in \text{Fact}(P)$.
Define

$$G \& H \triangleq G \cap H \quad \text{and} \quad G \oplus H \triangleq (G \cup H)^\perp.$$  

Observation 2.1.1:
Since $\text{Fact}(P)$ is a bounded lattice, the operations $\&$ and
$\oplus$ are associative and commutative, and admit identity
elements $T$ and $O$ respectively.

i.e. $T \& G = G$ and $O \oplus G = G$.  

Moreover, for all $G, H, K \in \text{Fact}(P)$,

$$(G \& H) \oplus (G \& K) \subseteq G \& (H \oplus K)$$

$$G \oplus (H \& K) \subseteq (G \oplus H) \& (G \oplus K).$$

Lemma 2.0.4 yields the de Morgan duality:

$$G \& H = (G^\perp \oplus H^\perp)^\perp$$ and $$G \oplus H = (G^\perp \& H^\perp)^\perp$$

for all $G, H \in \text{Fact}(P)$.

Note the verification conditions for $\&$ and $\oplus$:

(i) $1 \in G \& H$ if $1 \in G$ and $1 \in H$;

(ii) if $1 \in G$ or $1 \in H$ then $1 \in G \oplus H$, but the converse does not, in general, hold.

Lemma 2.1.2:

Fact$(P)$ is not necessarily a distributive lattice.

Proof:

There is a plentiful supply of finite phase spaces which provide counter-examples. Among the simplest is the monoid $P = \langle \mathbb{Z}/6\mathbb{Z}, +, 0 \rangle$ with $\bot_P = \{2, 5\}$.

$$\bot = \{0^\perp = \{n \in \mathbb{Z}/6\mathbb{Z} \mid n + 0 \equiv 2, 5 [\text{mod } 6]\} = \{2, 5\} = \{3\}^\perp,$$

$$\{1^\perp = \{n \in \mathbb{Z}/6\mathbb{Z} \mid n + 1 \equiv 2, 5 [\text{mod } 6]\} = \{1, 4\} = \{4\}^\perp,$$

$$\{2^\perp = \{n \in \mathbb{Z}/6\mathbb{Z} \mid n + 2 \equiv 2, 5 [\text{mod } 6]\} = \{0, 3\} = \{5\}^\perp.$$

$$\top = \{2^\perp \cap \{5^\perp = \{0, 3\}.$$ Let $G = \{1^\perp = \{4\}^\perp.$

**The pagination has gone askew; there are no pages 26, 27 and 28.**
We have examined all the duals of singletons, so 
\[ \text{Fact}(P) = \{ \top, \bot, G, T, O \} \] 
where \( T = \mathbb{Z}/6\mathbb{Z} \) and 
\[ O = T^\perp = \bigcap_{i < 6} \{ i \}^\perp = \emptyset. \]

The Hasse diagram for Fact(P) is as follows:

```
  T
 /\
/  \
G   \T
|   |
\|   \|
\0   \ \1
```

Hence \( G \& (1 \oplus \bot) = G \& T = G \)

While \( (G \& 1) \oplus (G \& \bot) = \emptyset \oplus \emptyset = \emptyset \)

---

**Lemma 2.1.3:**
Fact(P) is not necessarily a modular lattice, i.e. it may fail to satisfy the modularity condition:

\[ G \leq H \text{ then } G \oplus (H \& K) = H \& (G \oplus K) \]

(Of course, non-modularity implies non-distributivity, but this time our example isn't quite so simple.)

**Proof:** Consider the monoid \( P = <\mathbb{Z}/4\mathbb{Z}, +, 0> \) with \( \bot_P = \{ 1, 2 \} \)

\[
\top = \{ 0 \}^\perp = \{ n \in \mathbb{Z}/4\mathbb{Z} \mid n + 0 \equiv \{ 1, 2 \} \mod 4 \} = \{ 1, 2 \}
\]

\[
\{ 1 \}^\perp = \{ n \in \mathbb{Z}/4\mathbb{Z} \mid n + 1 \equiv \{ 1, 2 \} \mod 4 \} = \{ 0, 1 \}
\]

\[
\{ 2 \}^\perp = \{ n \in \mathbb{Z}/4\mathbb{Z} \mid n + 2 \equiv \{ 1, 2 \} \mod 4 \} = \{ 0, 3 \}
\]

\[
\{ 3 \}^\perp = \{ n \in \mathbb{Z}/4\mathbb{Z} \mid n + 3 \equiv \{ 1, 2 \} \mod 4 \} = \{ 2, 3 \}
\]
Now \( \mathbb{I} = \{15^+ \cap 25^+ = \{05\}, \quad T = \mathbb{Z}/4\mathbb{Z} \) and \( \mathbb{O} = \varnothing \).

Set \( \mathbb{G} = \{05^+ \cap 15^+ = \{15\} \),
\( \mathbb{H} = \{15^+ = \{0,1\} \),
\( \mathbb{K} = \{35^+ = \{2,3\} \),
\( \) and \( \mathbb{L} = \{25^+ = \{0,3\} \),
Then \( \mathbb{H}^+ = \mathbb{G} \),
\( \mathbb{K}^+ = \{25^+ \cap 35^+ = \{3\} \),
\( \) and \( \mathbb{L}^+ = \{05^+ \cap 35^+ = \{2\} \).

No three of the \( \{i5^+, \quad i < 4 \) have a non-empty intersection
So \( \text{Fact}(\mathbb{P}) = \{T, \mathbb{H}, \mathbb{L}, \mathbb{K}, \mathbb{L}, \mathbb{I}, \mathbb{G}, \mathbb{L}^+, \mathbb{K}^+, \mathbb{O} \} \).

The Hasse diagram should make things clearer.

Now \( \mathbb{G} \in \mathbb{H} \) but \( \mathbb{G} \perp (\mathbb{H} \wedge \mathbb{K}) = \mathbb{G} \perp \mathbb{O} = \mathbb{G} \),
While \( \mathbb{H} \perp (\mathbb{G} \perp \mathbb{K}) = \mathbb{H} \wedge T = \mathbb{H} \).

Exercise 2.1.4: Find a finite phase space such that \( \text{Fact}(\mathbb{P}) \) is non-modular and of smaller cardinality than the above.
Note 2.1.5:
Given that Fact(P) is a complete lattice, it is going to be fairly straightforward to give semantic definitions of (first-order) quantifiers for linear logic — they’ll just be infinite generalizations of the additives. If \( \Lambda \) is a formula in the language of predicate linear logic then its interpretation in \( P \), say \( s_\Lambda(A) \), will be a family of facts of \( P \), and

\[
\begin{align*}
  s_\Lambda(\land x. A) &= \land s_\Lambda(A) = \cap s_\Lambda(A) \\
  s_\Lambda(\lor x. A) &= \lor s_\Lambda(A) = (\cup s_\Lambda(A))^{++}.
\end{align*}
\]

The sequent calculus for predicate linear logic, call it \( LL \), is obtained from \( LL \) by suitably modifying the underlying language and adjoining the familiar quantifier rules:

\[
\begin{align*}
  \vdash A, \Delta & \quad (\land) \\
  \vdash A[t/x], \Delta & \quad (v)
\end{align*}
\]

provided \( x \) is not free in \( \Delta \)

Soundness and completeness of \( LL \) with respect to phase space semantics will come from tacking on the extra cases to the proofs for \( LL \). On this matter, we concur with Girard: the details are ‘...straightforward and boring and left to the reader!’ (Girard [1987], p.27)
2.2 The multiplicatives

Each of the multiplicative operations are definable in terms of the product operation (commonly used in work with monoids, semi-groups or groups):

if \( G, H \subseteq P \) then \( GH \triangleq \{ pq \mid p \in G \text{ and } q \in H \} \).

Since we are assuming \( P \) is a commutative monoid, \( GH = HG \). From the definition of \((-)^	ildeslash\), we have that \( G^\tildeslash G \subseteq \perp \) for all \( G \subseteq P \). Note, however, that even when \( G \) and \( H \) are both facts, \( GH \) is not necessarily a fact.

**Definition 2.2.1:**
Let \( G, H \in \text{Fact}(P) \).
Define

\[
G \otimes H \triangleq (GH)^	ildeslash\quad G \otimes H \triangleq (G^\tildeslash H^\tildeslash)^	ildeslash
\]

and \( G \circ H \triangleq (GH)^	ildeslash\).

**Lemma 2.2.2:** (Girard [1987])
The following properties are immediate consequences of the definitions:

For all \( G, H \in \text{Fact}(P) \),

\[
G \otimes H = (G^\tildeslash \otimes H^\tildeslash)^	ildeslash \quad G \otimes H = (G \circ H)^	ildeslash
\]

\[
G \otimes H = (G^\tildeslash \otimes H^\tildeslash)^	ildeslash \quad G \otimes H = (G \circ H)^	ildeslash
\]

\[
G \circ H = G^\circ \otimes H \quad G \circ H = G \circ H
\]
Lemma 2.2.3: (Girard [1981])

(i) The operations $\otimes$ and $\otimes$ are commutative and associative on $\text{Fact}(P)$, and admit identity elements $1$ and $\bot$ respectively,
i.e. $1 \otimes G = G$ and $1 \otimes G = G$.

(ii) The operation $\circ$ has the following properties with respect to $\text{Fact}(P)$:

(a) $G \circ H = H^\perp \rightarrow G^\perp$;

(b) $(G \otimes H) \circ K = G \circ (H \rightarrow K)$ and $G \circ (H \otimes K) = (G \circ H) \otimes K$;

(c) $1 \rightarrow G = G$ and $G \rightarrow \bot = G^\perp$.

Proof:
In virtue of the previous lemma, each part of (ii) is an immediate consequence of one of the parts of (i). The commutativity of $\otimes$ and $\otimes$ is also immediate. We show that, for any $G \in \text{Fact}(P)$, $1 \otimes G = G$, as follows.

First, observe that for any $G, H \in P$, if $1 \in H$ then $G \in GH$. Hence $G \subseteq 1G \subseteq (1G)^{11} = 1 \otimes G$.

Conversely, if $p \in 1$ and $q \in G$ then $pq \in G^{11} = G$.

Hence $1G \subseteq G$. But then $(1G)^{11} \subseteq G$, and we are done.

As a consequence, $1 \otimes G = G$.

All we now need show is that $\otimes$ is associative. To do this we show that

$(G \otimes H) \otimes K = ((G \otimes H) \otimes K)^{11} = (G \otimes (HK))^{11} = G \otimes (H \otimes K)$.
The middle identity comes from the associativity of the monoid operation, and the outer two are consequences of the following lemma.

**Lemma 2.2.4:** (Girard [1981])

Let $G, H \leq P$ be any subsets.

Then

$$G^{++} H^{++} \leq (GH)^{++}.$$ 

**Proof:**

Let $p \in G^{++}$, $q \in H^{++}$ and $r \in (GH)^{++}$. We need to show that $pqr \in \perp$. First fix $g \in G$. Then since $r \in (GH)^{++}$ we have that for all $h \in H$, $rg h \in \perp$. Hence $rg \in H^{++} = H^{++}$. Since $q \in H^{++}$, we have $rgq \in \perp$. Now $g \in G$ was arbitrary. So for all $g \in G$, $(qr)g \in \perp$. Hence $qr \in G^{++} = G^{++}$. Then since $p \in G^{++}$, we get $pqr \in \perp$. 

**Lemma 2.2.5:** (Girard [1981])

The following distributivity properties hold for all $G, H, K \in \text{Fact}(P)$:

\[\begin{align*}
\text{(i)} && G \otimes (H \oplus K) &= (G \otimes H) \oplus (G \otimes K), \\
& & G \otimes (H \& K) &= (G \otimes H) \& (G \otimes K), \\
& & (G \otimes H) \to K &= (G \to K) \& (H \to K), \\
& & G \to (H \& K) &= (G \to H) \& (G \to K).
\end{align*}\]

\[\begin{align*}
\text{(ii)} && 0 \otimes G &= 0, \\
& & T \& G &= T, \\
& & 0 \to G &= T, \\
& & G \to T &= T.
\end{align*}\]

\[\begin{align*}
\text{(iii)} && G \otimes (H \& K) &\leq (G \otimes H) \& (G \otimes K), \\
& & (G \& H) \oplus (G \& K) &\leq G \& (H \oplus K), \\
& & (G \& H) \oplus (G \& H) &\leq (G \& H) \to K, \\
& & (G \& H) \oplus (G \& H) &\leq G \to (H \oplus K).
\end{align*}\]

\[\begin{align*}
\text{(iv)} && G \otimes (H \& K) &\leq (G \otimes H) \& (G \otimes K).
\end{align*}\]
Proof: We prove one from each of (i), (ii) and (iii); the rest then follow by Lemma 2.2.2.

(i) \( G \otimes (H \oplus K) \subseteq (G \otimes H) \oplus (G \otimes K) \),
i.e. \( (G \cdot (H \cup K))^{\perp \perp} \subseteq ((G \cdot H)^{\perp \perp} \cup (G \cdot K)^{\perp \perp})^{\perp \perp} \).

Observe that \( G \cdot (H \cup K) \subseteq G \cdot H \cup G \cdot K \)
\[ \subseteq (G \cdot H)^{\perp \perp} \cup (G \cdot K)^{\perp \perp}. \]

By Lemma 2.2.4 \( G \cdot (H \cup K)^{\perp \perp} \subseteq (G \cdot (H \cup K))^{\perp \perp} \), since \( G = G^{\perp \perp} \).

Hence \( G \cdot (H \cup K)^{\perp \perp} \subseteq ((G \cdot H)^{\perp \perp} \cup (G \cdot K)^{\perp \perp})^{\perp \perp} \).

By Lemma 2.0.3(v), \( (G \cdot (H \cup K))^{\perp \perp} \subseteq ((G \cdot H)^{\perp \perp} \cup (G \cdot K)^{\perp \perp})^{\perp \perp} \).

(ii) \( G \otimes (H \& K) \subseteq (G \otimes H) \&(G \otimes K) \),
i.e. \( (G \cdot (H \& K))^{\perp \perp} \subseteq (G \cdot H)^{\perp \perp} \&(G \cdot K)^{\perp \perp} \).

Observe that \( G \cdot (H \& K) \subseteq G \cdot H \subseteq (G \cdot H)^{\perp \perp} \),
and \( G \cdot (H \& K) \subseteq G \cdot K \subseteq (G \cdot K)^{\perp \perp} \).

Hence \( G \cdot (H \& K) \subseteq (G \cdot H)^{\perp \perp} \&(G \cdot K)^{\perp \perp} \).

By Lemma 2.0.3(v), \( (G \cdot (H \& K))^{\perp \perp} \subseteq (G \cdot H)^{\perp \perp} \&(G \cdot K)^{\perp \perp} \).

Exercise 2.2.6: We've changed our mind: the proof of (iii) is left to the reader. (Write it out and you'll see why.)
The following lemma gives an illuminating characterization of $G \rightarrow o H$.

**Lemma 2.2.7:** (Girard [1987])

Let $G, H \in \text{Fact}(P)$. Then

$$G \rightarrow o H = \{ q \in P \mid \text{for all } p \in G \ (pq \in H) \}.$$ 

i.e. $G \rightarrow o H$ consists of those $q$ which send of all $G$ into $H$.

**Proof:**

Recall that $G \rightarrow o H \equiv (G H^\perp)^\perp$. Let $q$ be such that for all $p \in G$, $pq \in H$. Let $p \in G$ and $r \in H^\perp$, hence $pr \in G H^\perp$. Then $pq \in H = H^\perp$, hence $(pq)r = q(pr) \in \perp$. Hence $q \in (G H^\perp)^\perp$.

Conversely, suppose $q \in (G H^\perp)^\perp$. Fix $p \in G$. Then for all $r \in H^\perp$, $q(pr) = (pq)r \in \perp$. Hence $pq \in H^\perp = H$. 

**Corollary 2.2.8:**

Let $G, H \in \text{Fact}(P)$. Then $1 \in G \rightarrow o H$ iff $G \subseteq H$.

One would expect the inclusion relation on $\text{Fact}(P)$ to represent implication, but it is nice to have one's expectations confirmed. This corollary is used repeatedly in the proof of the soundness theorem.
To give an overview of all the operations on \( \text{Fact}(P) \), we summarize the verification conditions:

Let \( G, H \in \text{Fact}(P) \).

(i) \( 1 \in G^\perp \iff G \subseteq \perp \);
(ii) \( 1 \in G \circledast H \iff 1 \in G \) and \( 1 \in H \);
(iii) \( 1 \in G \oslash H \iff 1 \in G \) or \( 1 \in H \);*
(iv) \( 1 \in G \otimes H \iff 1 \in G \) and \( 1 \in H \);*
(v) \( 1 \in G \odot H \iff G^\perp \subseteq H \);
(vi) \( 1 \in G \downarrow \circ H \iff G \subseteq H \).

* Of course, the converses of (iii) and (iv) need not hold. The best available necessary and sufficient conditions are in terms of the duals:

(iii)' \( 1 \in G \oslash H \iff G^\perp \oslash H^\perp \subseteq \perp \)
(iv)' \( 1 \in G \otimes H \iff (GH)^\perp \subseteq \perp \)

Clause (v) is an admission that the only sensible way to think of \( G \circledast H \) is as \( G^\perp \circ H \).

**Lemma 2.2.9:**
For all \( G \in \text{Fact}(P) \),

(a) \( 1 \in G \circledast G^\perp \);
(b) \( 1 \in \perp \);
(c) \( 1 \in T \circledast G \).

**Proof:**
(a) \( 1 \in G \circledast G^\perp \iff G^\perp \subseteq G^\perp \). (b) has already been noted. For (c), recall that \( \perp \subseteq G \) for all \( G \in \text{Fact}(P) \), hence \( 1 \in T \circledast G \).
2.3 Soundness of LL

**Definition 2.3.0:**

Let $\text{Atom}(L)$ denote the set of all atomic formulae in the language $L$ of (propositional) linear logic. A phase structure for $L$ consists of a phase space $\langle P, \cdot, 1, \bot_p \rangle$ together with a function $S_p : \text{Atom}(L) \rightarrow \text{Fact}(P)$ satisfying

1. $S_p(A^\perp) = S_p(A)^\perp$ for each propositional letter $A$ of $L$;
2. $S_p(\bot) = \bot_p$ and $S_p(1) = 1^\perp$;
3. $S_p(T) = P$ and $S_p(\emptyset) = P^\perp$.

**Observation 2.3.1:**

Let $\langle P, \cdot, 1, \bot, S_p \rangle$ be a phase structure for $L$. Then the function $S_p$ has a unique extension \( \hat{S}_p : L \rightarrow \text{Fact}(P) \) satisfying

1. for all $A \in \text{Atom}(L)$, $\hat{S}_p(A) = S_p(A)$; and
2. for all $A, B \in L$,
   \[
   \begin{align*}
   \hat{S}_p(A \& B) &= \hat{S}_p(A) \& \hat{S}_p(B), \\
   \hat{S}_p(A \oplus B) &= \hat{S}_p(A) \oplus \hat{S}_p(B), \\
   \hat{S}_p(A \otimes B) &= \hat{S}_p(A) \otimes \hat{S}_p(B), \\
   \hat{S}_p(A \circ B) &= \hat{S}_p(A) \circ \hat{S}_p(B).
   \end{align*}
   \]

(We trust the notational abuses are not too distressing.)

**Definition 2.3.2:**

A formula $A \in L$ is valid in a phase structure $\langle P, \cdot, 1, \bot_p, S_p \rangle$ if $\hat{S}_p(A)$ is verified in $\langle P, \cdot, 1, \bot_p \rangle$. $A \in L$ is a linear tautology iff $A$ is valid in every phase structure $\langle P, \cdot, 1, \bot_p, S_p \rangle$. 
A sequence \( \Gamma \) of formulae of \( L \), say \( A_1, ..., A_n \), is valid in a phase structure \( \langle P, \cdot, \bot, \bot_P, S_P \rangle \) if \( S_P(A_1 \& \cdots \& A_n) \) is verified in \( \langle P, \cdot, \bot, \bot_P \rangle \). (We write \( S_P(\Gamma) \) for short.)

**Theorem 2.3.3:** (Girard [1987]).

The sequent calculus \( LL \) is sound with respect to validity in phases structures.

ie. if \( \Gamma \vdash \) provable in \( LL \)

then \( \Gamma \) is valid in every phase structure for \( L \).

**Proof:** by induction on a proof \( \pi \) of \( \Gamma \).

Let \( \langle P, \cdot, \bot, \bot_P, S_P \rangle \) be any phase structure for \( L \).

We use \( S \) as an abbreviation for \( S_P \).

(i) \( \pi \) is an instance of the identity axiom scheme, the \( \bot \) axiom, or the \( T \) axiom scheme.

By Lemma 2.2.9, \( 1 \in S(\Gamma) \).

(ii) \( \pi \) is obtained from \( \pi_0 \) and \( \pi_i \) by an application of (Cut):

\[
\frac{\Gamma, A, \Delta_0 \quad \Gamma, A, \Delta_i}{\Gamma, \Delta_0, \Delta_i} \quad \text{(Cut)}
\]

Let \( S(A) = G \), \( S(\Delta_0) = H \) and \( S(\Delta_i) = K \).

Suppose \( 1 \in G \otimes H \) and \( 1 \in G^+ \otimes K \). Then \( G^+ \leq H \), hence \( H^+ \leq G \), and \( G \leq K \). Hence \( H^+ \leq K \), so \( 1 \in H \otimes K \).

(iii) \( \pi \) is obtained from \( \pi_0 \) by an application of (Exch):

\[
\frac{\Gamma, \Delta}{\Gamma, \sigma(\Delta)} \quad \text{(Exch)}
\]

for some permutation \( \sigma \) of \( \Delta \).

By the commutativity of \( \otimes \), \( S(\Delta) = S(\sigma(\Delta)) \).
(iv) \( \Pi \) is obtained from \( \Pi_0 \) and \( \Pi_1 \) by an application of the rule (\&):

\[ \frac{\vdash A, \Delta \quad \vdash B, \Delta}{\vdash A \& B, \Delta} \] (\&)

Let \( s(A) = G \), \( s(B) = H \) and \( s(\Delta) = K \).
Suppose \( 1 \in G \& K \) and \( 1 \in H \& K \).
Then \( 1 \in (G \& K) \& (H \& K) = (G \& H) \& K \), by the distributivity of \& over \&.

(v) \( \Pi \) is obtained from \( \Pi_0 \) by an application of the rule (\texttt{Fst} \Theta):

\[ \frac{\vdash A, \Delta}{\vdash A \Theta B, \Delta} \] (\texttt{Fst} \Theta)

Let \( s(A) = G \), \( s(B) = H \) and \( s(\Delta) = K \).
Suppose \( 1 \in G \& K \). Then \( G^+ \subseteq K \). Since \( G^+ \cap H^+ \subseteq G^+ \),
we have \( (G \Theta H)^+ = G^+ \cap H^+ \subseteq K \).
Hence \( 1 \in (G \Theta H) \& K \).

(vi) \( \Pi \) is obtained from \( \Pi \), by an application of the rule (\texttt{Snd} \Theta):
Symmetric with case (v).

(vii) \( \Pi \) is obtained from \( \Pi_0 \) and \( \Pi_1 \) by an application of the rule (\Theta):

\[ \frac{\vdash A, \Delta_0 \quad \vdash B, \Delta_1}{\vdash A \Theta B, \Delta_0, \Delta_1} \] (\Theta)

Let \( s(A) = G \), \( s(B) = H \), \( s(\Delta_0) = K_0 \) and \( s(\Delta_1) = K_1 \).
Suppose \( 1 \in G \& K_0 \) and \( 1 \in H \& K_1 \).
Hence \( G^+ \subseteq K_0 \) and \( H^+ \subseteq K_1 \), so \( K_0^+ \subseteq G \) and \( K_1^+ \subseteq H \).
Hence \( K_0^+ \cdot K_1^+ \subseteq G \cdot H \subseteq (G \cdot H)^{++} = G \Theta H \).
By Lemma 2.0.3(v), \( (K_0^+ \cdot K_1^+)^{++} \subseteq G \Theta H \),
i.e. \( K_0^+ \Theta K_1^+ \subseteq G \Theta H \).
Hence \( (G \Theta H)^+ \subseteq K_0 \& K_1 \),
so \( 1 \in (G \Theta H) \& K_0 \& K_1 \).
(viii) \( \mathcal{T} \) is obtained from \( \mathcal{T}_0 \) by an application of the rule (\( \mathcal{B} \)):

\[
\frac{\vdash \Delta', \Delta}{\vdash \Delta', \Delta} \tag{\( \mathcal{B} \)}
\]

Immediate.

(ix) \( \mathcal{T} \) is obtained from \( \mathcal{T}_0 \) by an application of the rule (\( \mathcal{L} \)):

\[
\frac{\vdash \Delta}{\vdash \bot, \Delta} \tag{\( \mathcal{L} \)}
\]

Since \( \bot \) is the identity element for the operation \( \& \),

\[ s(\Delta) = \bot \& s(\Delta). \]

2.4 Completeness of \( \mathcal{LL} \)

For the purposes of the proof, we will work with a variant sequent calculus \( \mathcal{LL}^* \) obtained from \( \mathcal{LL} \) by omitting the exchange rule and adapting the axiom schemes and inference rules so as to be appropriate for multi-sets of formulae of \( \mathcal{L} \), i.e. we distinguish occurrences of formulae but ignore the order. Clearly, if \( \Delta \) is any multi-set of formulae of \( \mathcal{L} \), and \( \Delta' \) is any sequence of formulae of \( \mathcal{L} \) whose components are exactly the elements of \( \Delta \), then \( \vdash \Delta \) is provable in \( \mathcal{LL}^* \) iff \( \vdash \Delta' \) is provable in \( \mathcal{LL} \).

Multi-sets of formulae of \( \mathcal{L} \) form a commutative monoid: the operation is concatenation (denoted \( \ast \)) and the empty multi-set \( \emptyset \) is the identity element. We denote this monoid by \( \langle \mathbb{M}, \ast, \emptyset \rangle \). We will build a canonical phase structure \( \langle \mathbb{M}, \ast, \emptyset, \bot, \mathbb{M}, \mathbb{M} \rangle \).
which resembles the usual Lindenbaum algebra construction in the sense that formulae provably equivalent in LL will correspond to identical elements of \( \text{Fact}(M) \). This twist in this construction is the choice of \( \bot_m \).

**Theorem 2.4.0:** (Girard [1987])

The sequent calculus LL is complete with respect to validity in phase structures.

For any formula \( A \) of \( \mathcal{L} \),

if \( A \) is a linear tautology

then \( \vdash A \) is provable in LL.

**Proof.**

Consider the monoid \( \langle M, *, 1 \rangle \); we use \( \Delta, \Gamma, \Sigma \) to denote elements of \( M \). For the duration of the proof, we use \( \vdash_* \Delta \) as an abbreviation for \( \vdash \Delta \) is provable in LL*.

Now we define \( \bot_m = \{ \Delta \in M \mid \vdash_* \Delta \} \),

i.e. the interpretation of the multiplicative falsum is the set of all provable multi-sets!

So \( \langle M, *, \emptyset, \bot_m \rangle \) is a phase space; now we need a suitable interpretation function \( s_m \).

For each formula \( A \) of \( \mathcal{L} \), define

\[
Pr(A) = \{ \Gamma \in M \mid \vdash_* A, \Gamma \}
\]

(technically, we should write \( \vdash_* [\llbracket A \rrbracket] * \Gamma \), where the
square brackets \([ \cdot ]\) are for multi-sets what curly brackets \(\{ \cdot \}\) are for sets; we will only do so when it is necessary for clarity.)

**Lemma 2.4.1:**

For any formulae \(A \in L\) and \(B \in L\),

\[
\vdash_A A \rightarrow B \text{ iff } \text{Pr}(A) \subseteq \text{Pr}(B).
\]

Hence \(A \equiv L B \text{ iff } \text{Pr}(A) = \text{Pr}(B)\).

**Proof:**

Suppose \(\vdash_A A \rightarrow B\). Then we must have \(\vdash_A A^t, B^-\); i.e. in a cut-free proof, the last rule used must have been (8).

Let \(\Gamma \in \text{Pr}(A)\). Then \(\vdash_A \Gamma, A^-\). But \(\vdash_A \Gamma, A^-\) and \(\vdash_A A^t, B^-\) together imply \(\vdash_A B, \Gamma^-\) by (Cut). Hence \(\Gamma \in \text{Pr}(B)\).

Conversely, suppose \(\text{Pr}(A) \subseteq \text{Pr}(B)\). Now \(A^t \in \text{Pr}(A)\) since \(\vdash_A A, A^t\). Hence \(A^t \in \text{Pr}(B)\), so \(\vdash_B B, A^t\). Hence \(\vdash_A A \rightarrow B\) by the rule (8). \(\square\)

We don't actually need this lemma at the moment—we do need it later in proving the completeness of the modal sequent calculus LL! (Section 3.2)—but it is of help in providing motivation for the construction.

**Lemma 2.4.2:**

For each formula \(A \in L\), \(\text{Pr}(A) \in \text{Fact}(M)\).

**Proof:**

Consider the multi-set \([A] \in M\). The singleton set \(\{[A]\}\) has as its dual

\(\{A\}\)
$$\{ \{A\} \}^* = \{ \Gamma \in M \mid [A] \not\models \Gamma \}$$

$$= \{ \Gamma \in M \mid \not\vdash [A] \models \Gamma \}$$

$$= \text{Pr}(A).$$

Hence \( \text{Pr}(A) \in \text{Fact}(M) \). \( \square \)

**Lemma 2.4.3:** \( \text{Pr}(\bot) = \bot_M \) and \( \text{Pr}(T) = M \).

**Proof:**

Let \( \Delta \in \text{Pr}(\bot) \). Then \( \vdash \bot, \Delta \). We have \( \vdash T \), so by (Cut), \( \vdash \Delta \). Hence \( \Delta \in \bot_M \).

Conversely, let \( \Delta \in \bot_M \). Then \( \vdash \Delta \), hence \( \vdash \bot, \Delta \) by the rule (\( \bot \)). Hence \( \Delta \in \text{Pr}(\bot) \).

\( \text{Pr}(T) = \{ \Delta \in M \mid \vdash T, \Delta \} = M \), in virtue of the Axiom scheme for \( T \). \( \square \)

**Lemma 2.4.4:**

For each formula \( A \in L \), \( \text{Pr}(A^\uparrow) = \text{Pr}(A)^\downarrow \).

**Proof:**

\( \text{Pr}(A)^\uparrow = \{ \Delta \in M \mid \text{for all } \Gamma (\text{ if } \vdash A, \Gamma \text{ then } \vdash \Delta, \Gamma ) \} \)

Let \( \Delta \in \text{Pr}(A^\uparrow) \). Then \( \vdash A^\uparrow, \Delta \). Suppose \( \vdash A, \Gamma \) for \( \Gamma \in M \).

Then by (Cut), \( \vdash \Delta, \Gamma \). Hence \( \Delta \in \text{Pr}(A)^\downarrow \).

Conversely, suppose \( \Delta \in \text{Pr}(A)^\downarrow \). Now \( \vdash A, A^\uparrow \) hence \( \vdash \Delta, A^\uparrow \). So \( \Delta \in \text{Pr}(A^\uparrow) \). \( \square \)
Now set \( S_m = \text{Pr} \, \Gamma \text{Atom}(L) \). The previous two lemmas together imply that the function \( S_m : \text{Atom}(L) \to \text{Fact}(M) \) satisfies the conditions necessary for \( \langle M, \ast, \emptyset, \bot_m, S_m \rangle \) to be a phase structure.

We are not quite finished: we still have to check that \( \hat{S}_m = \text{Pr} \). Otherwise put, we have to show that \( \text{Pr} : L \to \text{Fact}(M) \) is an homomorphism, considering \( L \) and \( \text{Fact}(M) \) as algebras of the same type.

**Lemma 2.4.5:**

For all \( A \in L, B \in L \)

(i) \( \text{Pr}(A \& B) = \text{Pr}(A) \& \text{Pr}(B) \);

(ii) \( \text{Pr}(A \oplus B) = \text{Pr}(A) \oplus \text{Pr}(B) \);

(iii) \( \text{Pr}(A \otimes B) = \text{Pr}(A) \otimes \text{Pr}(B) \);

(iv) \( \text{Pr}(A \& B) = \text{Pr}(A) \& \text{Pr}(B) \).

**Proof:**

We give proofs of (i) and (iii); (ii) and (iv) follow by Lemma 2.4.4 together with the definitions of \((A \oplus B)^\perp\) and \((A \otimes B)^\perp\) for formulae \( A \) and \( B \) of \( L \).

(i) Observe that \( \vdash A \& B, \Delta \) iff \( \vdash A, \Delta \) and \( \vdash B, \Delta \)

Hence \( \Delta \in \text{Pr}(A \& B) \) iff \( \Delta \in \text{Pr}(A) \cap \text{Pr}(B) \).

(iii) Let \( \Pi \in \text{Pr}(A \otimes B) \) and fix some \( \Xi \in (\text{Pr}(A), \text{Pr}(B))^\perp \).

Then for all \( \Delta_0 \) and \( \Delta_1 \), if \( \vdash A, \Delta_0 \) and \( \vdash B, \Delta_1 \)

then \( \vdash A_0, \Delta_1, \Xi \). Consider the case of \( \Delta_0 = \lbrack A^\perp \rbrack \)

and \( \Delta_1 = \lbrack B^\perp \rbrack \). Since \( \vdash A, A^\perp \) and \( \vdash B, B^\perp \),
We get $\Gamma \vdash A^*, B^*, \Sigma$. Hence $\Gamma \vdash (A \otimes B)^*, \Sigma$ by the (S) rule. Now $\Gamma \in \text{Pr}(A \otimes B)$ so $\Gamma \vdash A \otimes B, \Gamma$. By (cut), we get $\Gamma \vdash \Gamma, \Sigma$.
Hence $\Gamma \in (\text{Pr}(A) \otimes \text{Pr}(B))^{**} = \text{Pr}(A) \otimes \text{Pr}(B)$.

Conversely, suppose $\Delta_0 \in \text{Pr}(A)$ and $\Delta_1 \in \text{Pr}(B)$.
Then $\Gamma \vdash A, \Delta_0$ and $\Gamma \vdash B, \Delta_1$. By the rule (⊗), we have $\Gamma \vdash A \otimes B, \Delta_0, \Delta_1$. Hence $\Delta_0 \otimes \Delta_1 \in \text{Pr}(A \otimes B)$.
Hence $\text{Pr}(A) \otimes \text{Pr}(B) \subseteq \text{Pr}(A \otimes B)$, and so $\text{Pr}(A) \otimes \text{Pr}(B) \subseteq \text{Pr}(A \otimes B)$.

And now for the punchline...

Let $A$ be any linear tautology; i.e., $A$ is valid in every phase structure. In particular, $A$ is valid in $\langle M, *, \varnothing, \bot, \text{Pr} \rangle$. So $\varnothing \in \text{Pr}(A)$, hence $\Gamma \vdash A$.
Hence $\vdash A$ is provable in $\text{LL}$. ■
3. The exponential (modal) operators.

By enriching propositional linear logic with modal, or 'exponential', connectives, we recover expressive power; in particular, sufficient expressive power to be able to interpret intuitionistic logic in the resulting system (Section 3.3). Girard's slogan is that 'usual [intuitionistic] logic is obtained from [propositional] linear logic by a passage to the limit'; the mathematical content of this slogan is a result called the Approximation Theorem. (Section 4.1).

Girard calls the modalities 'of course' (denoted '!'') and 'Why not' (denoted '?') their behaviour is similar to that of necessity and possibility, respectively, in the classical modal logic S4, modulo the weirdness of the underlying propositional logic. (Girard's rationale for the non-standard notation is that he doesn't want linear logic dismissed as 'yet another modal system': Girard [1987], p.27.)

One of the crucial aspects of this extension of linear logic is that the structural rules of contraction and thinning re-appear, but are allowed only in modal contexts.

Why the term 'exponential'? (Pourquoi pas?) One answer is that what else will fill the blank in the sequence 'additive, multiplicative, ___': (Bien sur!) An explanation is that the modal contraction rule results in exponential growth in the size of proof.
nets under normalization (elimination of cuts): see section 6.4. Also, the exponential ! is connected with the ‘internal hom’ or ‘exponential’ functor \[ \_ \Rightarrow \_ : C^\circ \times C \rightarrow C \] in a Cartesian closed category \( C \); i.e. intuitionistic implication: see sections 5.1 and 5.2.

3.0 The sequent calculus \( \text{LL}! \)

Definition 3.0.0:
The language \( L! \) of modal linear logic is defined in the same way as the language \( L \) (Definition 1.1.0), with the additional clause:
- If \( A \) is a formula of \( L! \), then so are \( !A \) and \( ?A \).

The expression \( A^\perp \), where \( A \) is a formula of \( L! \), is defined in the same way as it is for formulae of \( L \) (Definition 1.1.1), with the additional clause:
- If \( A \) is a formula of \( L! \),
  \( (!A)^\perp \defeq (?A^\perp) \) and \( (?A)^\perp \defeq (!A^\perp) \).

Not unexpectedly, de Morgan duality strikes again. Girard does not give names to his sequent calculi: we have chosen the names \( \text{LL} \) and \( \text{LL}! \) for their obviousness. (the name \( \text{LLM} \) is used in Danos and Regnier [1989] — for the same system we call \( \text{LLM} \))
Definition 3.0.1:
The sequent calculus \( \mathbf{LL}! \) consists of the same axiom schemes and inference rules as \( \mathbf{LL} \), deemed now to apply to \( \mathcal{L}! \)-sequents, together with the following inference rules:

\[
\begin{align*}
\frac{\vdash \Delta}{\vdash ?\Delta, \Delta} & \quad (\text{Th?}) \\
\frac{\vdash ?\Delta, ?A, \Delta}{\vdash ?\Delta, \Delta} & \quad (\text{c?}) \\
\frac{\vdash ?A, \Delta}{\vdash ?\Delta, \Delta} & \quad (D?) \\
\frac{\vdash ?A, \Delta}{\vdash A, ?\Delta} & \quad (!) \\
\frac{\vdash A, ?\Delta}{\vdash A, ?\Delta} & \quad (!) \\
\frac{\vdash A, ?\Delta}{\vdash A, ?\Delta} & \quad (!)
\end{align*}
\]

If \( \Delta \) is the sequence \( B_1, \ldots, B_n \) then \( ?\Delta \) is an abbreviation for \( ?B_1, \ldots, ?B_n \).

Girard uses the name \( (W?) \) instead of our \( (\text{Th?}) \), i.e. 'Weakening' rather than 'thinning'. We have changed names because in the relevance logic tradition, \( (W) \) is the name of an axiom scheme (in a Hilbert-style system) for \textit{contraction} — the nomenclature derives from Combinatory logic; in Smullyan-iian terms, 'W' is the 'warblers'. The name \( (D?) \) is short for 'dereliction': 'who cares' about \( A \)?

Again, we are safe to restrict the identity axiom scheme to propositional letters and their duals:

\[
\begin{align*}
\frac{\vdash A, A^\dagger}{\vdash A, ?(A^\dagger)} & \quad (D?) \\
\frac{\vdash A, ?(A^\dagger)}{\vdash A, A^\dagger} & \quad (!) \\
\frac{\vdash ?A, A^\dagger}{\vdash ?A, A^\dagger} & \quad (D?) \\
\frac{\vdash A^\dagger}{\vdash A^\dagger} & \quad (!) \\
\frac{\vdash ?A, A^\dagger}{\vdash A^\dagger} & \quad (!)
\end{align*}
\]

(We omit exchanges.)
The modal connectives provide a rather intriguing linkage between the additives and the multiplicatives.

**Lemma 3.0.2: (Girard [1987])**

The following are provable equivalences in LL!: 

(i)  \( ! (A \& B) \equiv_u ! (A) \otimes ! (B) \);
(ii) \( ? (A \oplus B) \equiv_u ? (A) \otimes ? (B) \);
(iii) \( ! \top \equiv_u \bot \);
(iv) \( ? \bot \equiv_u \bot \).

**Proof:** We exhibit proofs for (ii) and (iv); (i) and (iii) then follow by duality. Exchanges are, as usual, omitted.

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<td>( \vdash \bot i )</td>
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3.1 Topolinear spaces

A semantics for modal linear logic is obtained from the phase space semantics by adding a bit more apparatus. The apparatus in question resembles a topology on \( \text{Fact}(P) \) (but is not quite one). We will have closed facts \( (?G) \) and open facts \( (!G) \). The closedness in this sense is quite different from closure in the sense of the closure operator \( (\neg)^{++} \); all elements of \( \text{Fact}(P) \) are closed in the latter sense. (Maybe this is why Girard makes no mention of \( (\neg)^{++} \) as a closure operator. We prefer to call a closure operator a closure operator.)

**Definition 3.1.0:**

A **topolinear space** is a quintuple \( <P, \cdot, 1, \bot, \mathcal{F}> \) where \( <P, \cdot, 1, \bot> \) is a phase space and \( \mathcal{F} \subseteq \text{Fact}(P) \) is a family of facts of \( P \) satisfying the following conditions:

(i) for any \( X \in \mathcal{F} \), \( \bigwedge X \in \mathcal{F} \); in particular, \( T \in \mathcal{F}, (T = P) \);

(ii) if \( F_1, \ldots, F_n \in \mathcal{F} \) then \( F_1 \circ \cdots \circ F_n \in \mathcal{F} \);

(iii) \( \bigcap \mathcal{F} = \bot \);

(iv) if \( F \in \mathcal{F} \) then \( F \circ S \mathcal{F} = F \). (\( S \) is idempotent on \( \mathcal{F} \))
$G \in \text{Fact}(P)$ is a \textit{closed fact} iff $G \in \text{IF}$.

$G \in \text{Fact}(P)$ is an \textit{open fact} iff $G^+ \in \text{IF}$.

As before, $G \in \text{Fact}(P)$ is \textit{verified} in $\langle P, \cdot, 1, \bot, \text{IF} \rangle$ iff $1 \in G$.

Let $G \in \text{Fact}(P)$.

The \textit{affirmation} of $G$, $!G$ (of course $G$) is defined as follows:

$$!G \overset{\text{df}}{=} (U \{ H \mid H \subseteq G \text{ and } H^+ \in \text{IF} \})^\bot$$

The \textit{consideration} of $G$, $?G$ (why not $G$) is defined as follows:

$$?G \overset{\text{df}}{=} \bigcap \{ F \mid G \subseteq F \text{ and } F \in \text{IF} \}.$$

\underline{Lemma 3.1.1}:

Let $G \in \text{Fact}(P)$.

(i) $(!G)^+ = (?G)^*$ and $(!G)^* = (?G)^+$

(ii) $!G$ is the largest (w.r.t. $\leq$) open fact included in $G$, and $?G$ is the smallest (w.r.t. $\leq$) closed fact containing $G$.

\underline{Proof}:

(i) $(!G)^+ = \left( \bigcup \{ H \mid H \subseteq G \text{ and } H^+ \in \text{IF} \} \right)^\bot$

$= \bigcap \{ H^+ \mid H \subseteq G \text{ and } H^+ \in \text{IF} \}$ by Lemma 2.0.4,

$= \bigcap \{ H^+ \mid G^+ \subseteq H^+ \text{ and } H^+ \in \text{IF} \}$

$= (?G)^*$.

$(?G)^+ = (!G)^*$ similarly.

(ii) follows from (i) and the definitions.
So for all \( G \in \text{Fact}(P) \), \( 1 \in \neg G \rightarrow G \) and \( 1 \in G \rightarrow \neg G \).

Also,
- \( G \) is closed iff \( G = \neg F \) for some \( F \in \mathcal{F} \);
- \( G \) is open iff \( G = \neg (F^+) \) for some \( F \in \mathcal{F} \).

**Lemma 3.1.2:**
For all \( G \in \text{Fact}(P) \), \( G = \neg G \) and \( \neg \neg G = G \).

**Proof:** We have \( G = \neg G \).

Now \( G = \bigcup \{ H \mid H \in \neg G \text{ and } H^+ \in \mathcal{F} \}^{\perp} \).

Since \( (G^+)^+ \in \mathcal{F} \), we must have \( G \subseteq \neg G \).

\( \neg G = G \) follows by duality.

**Lemma 3.1.3:**
\( \neg T = 1 \) and \( \neg \bot = \bot_p \).

**Proof:**
\begin{align*}
\neg T &= \bigcup \{ H \mid H \in T \text{ and } H^+ \in \mathcal{F} \}^{\perp} \\
&= \bigcup \{ H \mid H \in \neg \mathcal{F} \}^{\perp} \\
&= \bigcap \{ H^+ \mid H \in \mathcal{F} \}^{\perp}, \text{ by Lemma 2.0.4.} \\
&= (\bigcap \mathcal{F})^{\perp} \\
&= (\neg \bot)^{\perp}, \text{ by Condition (iii) on } \mathcal{F}.
\end{align*}

**Corollary 3.1.4:** If \( 1 \in \neg G \) then \( \neg G = 1 \).
Hence \( 1 \) is the sum (supremum) of all affirmations, (Which sounds better than '\( 1 \) is the intersection of all considerations'.)

The corresponding syntactic result is readily verified:
if \( \neg T \neg A \) is provable in \( \mathcal{L} \), then \( \neg A \equiv u. \ 1 \).
If we set \( \mathcal{G} = \{ G \in \text{Fact}(P) \mid G^\perp \in \mathcal{F} \} \) then we get a list of properties of open facts just by dualizing the conditions on \( \mathcal{F} \):

(i) for any \( Y \in \mathcal{G} \), \( (\cup Y)^\perp \in \mathcal{G} \);
(ii) if \( G_1, \ldots, G_n \in \mathcal{G} \), then \( G_1 \otimes \ldots \otimes G_n \in \mathcal{G} \);
(iii) \( (\cup \mathcal{G})^\perp = \mathcal{1} \);
(iv) if \( G \in \mathcal{G} \) then \( G \otimes G = G \).

In summary, we have the following verification conditions:

for any \( G \in \text{Fact}(P) \),

1. \( 1 \in !G \) iff \( !G = \mathcal{1} \);
2. \( 1 \in ?G \) iff for all \( F \supseteq G \) with \( F \in \mathcal{F} \), \( 1 \in F \).

The general picture of a topolinear \( \text{Fact}(P) \) is as follows (cf. the picture following Examples 2.0.7).

![Diagram of open facts (\( \mathcal{G}_o \)), clopen facts, closed facts (\( \mathcal{F} \)), and the duality relationship between open and closed facts.]
3.2 Soundness and Completeness of \( \text{LL}_! \)

**Definition 3.2.0:**

A topolinear structure for \( \text{LL}_! \) is a sextuple \( \langle P, \cdot, 1, \bot, F, s_p \rangle \) where \( \langle P, \cdot, 1, \bot, F \rangle \) is a topolinear space and \( s_p : \text{Atom}(\text{LL}_!) \to \text{Fact}(P) \) is a function satisfying the same three conditions as in Definition 2.3.0. (Note that \( \text{Atom}(\text{LL}_!) = \text{Atom}(\mathcal{L}) \).)

The function \( s_p \) has a unique extension \( \hat{s}_p : \mathcal{L}_! \to \text{Fact}(P) \) which, in addition to respecting \( \& \), \( \oplus \), \( \otimes \) and \( \otimes_0 \) (as in Observation 2.3.1), satisfies, for all formulae \( A, B \in \mathcal{L}_! \),
\[ \hat{s}_p(\not A) = \not \hat{s}_p(A) \quad \text{and} \quad \hat{s}_p(\exists A) = \exists \hat{s}_p(A). \]

A formula \( A \) of \( \mathcal{L}_! \) is valid in a topolinear structure \( \langle P, \cdot, 1, \bot, F, s_p \rangle \) iff \( 1 \in \hat{s}_p(A) \).

A sequence \( \Gamma \) of formulae of \( \mathcal{L}_! \), say \( A_n \ldots A_1 \), is valid in a topolinear structure \( \langle P, \cdot, 1, \bot, F, s_p \rangle \) iff \( 1 \in \hat{s}_p(A_{\Gamma}; \otimes_\otimes \ldots \otimes_\otimes A_1) \); \( \hat{s}_p(\Gamma) \) for short.

**Theorem 3.2.1:** (Girard [1987])
The sequent calculus \( \text{LL}_! \) is sound with respect to validity in topolinear structures.

**Proof:** by induction on a proof \( \Pi \) in \( \text{LL}_! \) of \( \Gamma \).
Let \( \langle P, \cdot, 1, \bot, s_p \rangle \) be any topolinear structure for \( \mathcal{L}_! \).
Again, \( s \) is an abbreviation for \( \hat{s}_p \).

The first nine cases of the induction are as in the proof of Theorem 2.3.3.
(x) \( T \) is obtained from \( T \) by an application of the rule (Th?):

\[
\frac{\vdash \Delta}{\vdash \square\text{?}A, \Delta} \quad \text{(Th?)}
\]

Let \( s(A) = F \) and \( s(\Delta) = G \). Assume \( 1 \in G \). Then \( G^+ \subseteq \bot \).

But since \( \square\text{?}F \subseteq \square \), we have \( \bot \subseteq \square\text{?}F \). Hence \( G^+ \subseteq \square\text{?}F \),
so \( 1 \in G \& \square\text{?}F = \square\text{?}F \& G \).

(xi) \( T \) is obtained from \( T \) by an application of the rule (C?):

\[
\frac{\vdash \square\text{?}A, \square\text{?}A, \Delta}{\vdash \square\text{?}A, \Delta} \quad \text{(C?)}
\]

Let \( s(A) = F \) and \( s(\Delta) = G \). Assume \( 1 \notin \square\text{?}F \& \square\text{?}F \& G \).

Since \( \square\text{?}F \subseteq \square \), we have \( \square\text{?}F \& \square\text{?}F = \square\text{?}F \) (idempotence of \& on \&). Hence \( 1 \in \square\text{?}F \& G \).

(xii) \( T \) is obtained from \( T \) by an application of the rule (D?):

\[
\frac{\vdash A, \Delta}{\vdash \square\text{?}A, \Delta} \quad \text{(D?)}
\]

Let \( s(A) = F \) and \( s(\Delta) = G \). Assume \( 1 \in F \& G \). Hence \( F^+ \subseteq G \).

But \( \square\text{!}(F^+) \subseteq F^+ \), hence \( \square\text{!}(F^+) \subseteq G \). So \( 1 \in \square\text{?}F \& G \).

(xiii) \( T \) is obtained from \( T \) by an application of the rule (!):

\[
\frac{\vdash A, \square\text{?}B_i, \ldots, \square\text{?}B_n}{\vdash \square\text{!}A, \square\text{?}B_i, \ldots, \square\text{?}B_n} \quad \text{(!)}
\]

Let \( s(A) = G \) and \( s(B_i) = F_i \). \( i = 1, \ldots, n \).

Assume \( 1 \in G \& \square\text{?}F_1 \& \ldots \& \square\text{?}F_n \). Hence \( G^+ \subseteq (\square\text{?}F_1 \& \ldots \& \square\text{?}F_n) \).

Now \( \square\text{?}F_1 \& \ldots \& \square\text{?}F_n \) is closed since each \( \square\text{?}F_i \) is closed,
and \( \square\text{?}(G^+) \) is the smallest closed fact containing \( G^+ \),
so \( \square\text{?}(G^+) \subseteq (\square\text{?}F_1 \& \ldots \& \square\text{?}F_n) \).

Hence \( 1 \in \square\text{!}G \& (\square\text{?}F_1 \& \ldots \& \square\text{?}F_n) \).
Theorem 3.2.2: (Girard [1987])

The sequent calculus $LL!$ is complete with respect to validity in topolinear structures.

Proof:
We adapt the proof of completeness of $LL$. Recall the phase structure $\langle M, \ast, \emptyset, \bot \rangle$, where

$$\bot = \{ \Delta \in M \mid \vdash^* \Delta \}. \tag{1}$$

In this context, "$\vdash^* \Delta"$ is an abbreviation for "$\vdash \Delta$ is provable in $LL!*"$, where $LL!*$ is the multi-set version of $LL!$. As before, we define the function $Pr : L! \rightarrow \text{Fact}(M)$ by

$$Pr(A) = \{ \Gamma \in M \mid \vdash^* A, \Gamma \}. \tag{2}$$

Now set $E = \{ Pr(\gamma A) \mid A \in L! \}$
and $IF = \{ \cap X \mid X \subseteq E \}$.

To complete the proof, we need to show that

1. $IF$ satisfies conditions (i)-(iv) of Definition 3.1.0, and hence $\langle M, \ast, \emptyset, \bot, IF \rangle$ is a topolinear space;
2. $Pr(\gamma A) = \gamma Pr(A)$
and $Pr(! A) = ! Pr(A)$ for all $A \in L!$.

Recall Lemma 2.4.1. In this context, we have the result: $A \equiv_{LL!} B$ iff $Pr(A) = Pr(B)$.

Exercise 3.2.3:
Verify the provable equivalence $\gamma A \equiv_{LL!} \gamma A \otimes \gamma A$. 

Lemma 3.2.4: (Girard [1981])

The set \( E = \{ \text{Pr}(?A) \mid A \in L \} \) satisfies the following conditions:

(ii) \( E \) is closed with respect to finite \( \& \)'s;

(iii) \( \bot_m = \cap E \).

(iv) for all \( \text{Pr}(?A) \in E \), \( \text{Pr}(?A) = \text{Pr}(?A) \& \text{Pr}(?A) \).

Proof:

(ii) Recall that \( (?A) \& (?B) \equiv \text{u} (?A \ominus B) \). (Lemma 3.0.2).

Then \( \text{Pr}(?A) \& \text{Pr}(?B) = \text{Pr}( (?A) \& (?B) ) = \text{Pr}( (?A \ominus B) ) \in E. \)

By the obvious induction, \( E \) is closed with respect to finite \( \& \)'s.

(iii) \( \text{Pr}(?\emptyset) \in E \) and \( \text{Pr}(\bot) = \text{Pr}(?\emptyset) \) since \( \bot \equiv \text{u} ?\emptyset \).

Hence \( \bot_m = \text{Pr}(\bot) \in E. \)

Let \( \Gamma \in \bot_m \). Then \( \Gamma \vdash \Gamma \), hence \( \Gamma \vdash \Gamma, ?A \) by the rule (th?). Hence \( \Gamma \in \text{Pr}(?A) \).

Thus \( \bot_m \subseteq \text{Pr}(?A) \) for all \( \text{Pr}(?A) \in E \),

hence \( \bot_m = \cap E. \)

(iv) Since \( ?A \equiv \text{u} (?A \& ?A) \),

\( \text{Pr}(?A) = \text{Pr}( ?A \& ?A) = \text{Pr}(?A) \& \text{Pr}(?A). \)

Now if \( F = \{ \cap \cap X \mid X \in E \} \)

then clearly

(i) \( F \) is closed with respect to arbitrary intersections;

and (iii) \( \bot_m = \cap F. \)
To verify that $\mathcal{F}$ satisfies the remaining two conditions, we need the following result.

**Lemma 3.2.5**: (Girard [1981])

Let $\langle P, \cdot, 1, \perp_P \rangle$ be any phase space and let $\{G_i\}_{i \in \mathcal{I}}$ and $\{H_j\}_{j \in \mathcal{J}}$ be families of facts of $P$.

Then

$$\bigcap_{(i,j) \in \mathcal{I} \times \mathcal{J}} (G_i \& H_j) = \left( \bigcap_{i \in \mathcal{I}} G_i \right) \& \left( \bigcap_{j \in \mathcal{J}} H_j \right)$$

i.e. $\&$ distributes over arbitrary intersections.

**Proof:**

$p \in \bigcap_{i,j} (G_i \& H_j)$

iff $p \in \bigcap_{i,j} (G_i^\perp \cdot H_j^\perp)$

iff $\forall i \in \mathcal{I}, \forall j \in \mathcal{J}, \forall q \in G_i, \forall r \in H_j \quad (pqr \in \perp)$

iff $\forall i \in \mathcal{I}, \forall q \in G_i^\perp, \forall j \in \mathcal{J}, \forall r \in H_j^\perp \quad (pqr \in \perp)$

iff $\forall i \in \mathcal{I}, \forall q \in (\bigcap_{i} G_i)^\perp, \forall j \in \mathcal{J}, \forall r \in (\bigcap_{j} H_j)^\perp \quad (pqr \in \perp)$

iff $\forall i \in \mathcal{I}, \forall q \in (\bigcap_{i} G_i)^\perp \cdot (\bigcap_{j} H_j)^\perp \quad (pqr \in \perp)$

iff $p \in ((\bigcap_{i} G_i)^\perp \cdot (\bigcap_{j} H_j)^\perp)$

iff $p \in \left( \bigcap_{i} G_i \right) \& \left( \bigcap_{j} H_j \right)$

The equivalence of lines 5 and 6 is due to Lemma 2.0.4. □
(ii) \( \mathcal{B} \) is closed with respect to finite \( \mathcal{B} \)'s since

\[
\left( \bigcap_{i \in I} \Pr(\neg A_i) \right) \mathcal{B} \left( \bigcap_{i \in J} \Pr(\neg B_i) \right) = \bigcap_{(i, j) \in I \times J} \Pr(\neg (A_i \oplus B_j))
\]

and (iv) \( \mathcal{B} \) is idempotent on \( \mathcal{L} \):

\[
\left( \bigcap_{i \in I} \Pr(\neg A_i) \right) \mathcal{B} \left( \bigcap_{i \in I} \Pr(\neg A_i) \right) = \bigcap_{i \in I} \Pr(\neg A_i)
\]

We have now established that \( \langle M, *, \emptyset, \bot, \mathcal{L}, \mathcal{F} \rangle \) is a topolinear space.

Our remaining task is the verification of \( \Pr(A) = \Pr(\neg A) \); since \( \Pr(A^+) = \Pr(A)^+ \) (Lemma 2.4.4), we can readily derive \( \Pr(A) = \Pr(\neg A) \).

**Lemma 3.2.6:** (Girard [1987])

\( \vdash A \rightarrow \neg B \) is provable in LL! if and only if \( \vdash \neg \neg A \rightarrow \neg B \) is provable in LL!.

**Proof:**

If \( \vdash A \rightarrow \neg B \) is provable in LL!, then so is \( \vdash A^+, \neg B \).

\[
\vdash A^+, \neg B \quad (1)
\]

\[
\vdash \neg \neg (A^+), \neg B \quad (8)
\]

\[
\vdash \neg \neg A \rightarrow \neg B
\]
Conversely, if \( \vdash ?A \rightarrow ?B \) is provable in \( \text{LL}! \) then so is \( \vdash ! (A^\dagger) \rightarrow ?B \).

\[
\begin{align*}
\vdash A, A^\dagger & \quad \vdash ?A, A^\dagger \quad \vdash ! (A^\dagger), ?B \\
\vdash ?A, A^\dagger & \quad \vdash ! (A^\dagger), ?B \\
\vdash A^\dagger, ?B & \quad \vdash A \rightarrow ?B
\end{align*}
\]

Now, \( ? \Pr (A) \) is the smallest fact in \( \text{IF} \) containing \( \Pr (A) \).

\[
\begin{align*}
? \Pr (A) &= \cap \{ \cap X \mid \Pr (A) \subseteq \cap X \text{ and } X \subseteq E \} \\
&= \cap \{ \cap \Pr (?B) \mid \Pr (A) \subseteq \Pr (?B) \} \\
&= \cap \{ \Pr (?B) \mid \vdash \ast A \rightarrow ?B \} \\
&= \cap \{ \Pr (?B) \mid \vdash ?A \rightarrow ?B \} \\
&= \cap \{ \Pr (?B) \mid \Pr (?A) \subseteq \Pr (?B) \} \\
&= \Pr (?A).
\end{align*}
\]

3.3 A translation of intuitionistic logic

In \( \text{LL}! \), the modal versions of contraction and thinning yield as theorems

\[
! A \rightarrow ( ! A \otimes ! A) \quad \text{and} \quad ( ! A \otimes ! B) \rightarrow ! A
\]

We can think of occurrences of \( 
\) and \( ? \) which are inside the scope of one of the binary propositional connectives as markers indicating that contraction and/or thinning may have been used. (Cut-free derivations of \( ( ! A) \delta ( ? (A^\dagger)) \) are an exception.) Since the sequent calculus for
intuitionistic logic, $\text{LJ}$, has contraction and thinning among its inference rules, one would expect that a translation of intuitionistic logic into modal linear logic would make heavy use of the modalities. (cf. the translation of intuitionistic logic into the classical modal logic $\mathbf{S}4$) the following translation does make heavy use of the modalities, but not uniformly.

Definition 3.3.0:
Let $\text{L}_I$ be a language for intuitionistic propositional logic, generated from propositional letters and constants $t$ and $f$ using the unary connective $\neg$ and the binary connectives $\land, \lor$ and $\Rightarrow$.

We define a translation function $(\cdot)^\circ : \text{L}_I \rightarrow \text{L}_I$ by induction on formulae as follows:

\[
\begin{align*}
(A^\circ) &= A & \text{if } A \text{ is a propositional letter,} \\
(t^\circ) &= T \\
(f^\circ) &= 0 \\
(A \land B)^\circ &= A \land B \\
(A \lor B)^\circ &= !(A^\circ) \lor !(B^\circ) \\
(A \Rightarrow B)^\circ &= !(A^\circ) \rightarrow B^\circ \\
(\neg A)^\circ &= !(A^\circ) \rightarrow 0
\end{align*}
\]

The mapping can be extended to a language for intuitionistic predicate logic, with quantifiers $\forall$ and $\exists$, as follows:

\[
\begin{align*}
(\forall x. A)^\circ &= \land x. A^\circ \\
(\exists x. A)^\circ &= \lor x. !(A^\circ)
\end{align*}
\]
We will concentrate on the translation for propositional logics; keen readers can supply the extra details themselves.

At the moment, the most we can say about the virtue of this translation is that it works. In fact, it works very well: the translation is faithful, not only with respect to provable formulae but also with respect to proofs; we prove this below. An explanation as to why the translation works is to be found in coherent space semantics (Section 5).

Note that the translation of intuitionistic negation involves the additive falsum $\bot$ rather than the multiplicative falsum $\perp$. If we had

\[(\neg A)^0 = ! (A^0) \rightarrow \bot\]

then \[ (\neg A)^0 \equiv_{\text{ll}} (\neg (A^0))^1. \]

Girard's comment is that intuitionistic negation has always been widely criticized (Girard [1987], p.79).

There is a formal analogy between the above translation and the definition of Kleene's 'slash' for formulae of Heyting Arithmetic. (Girard attributes this observation to Andrej Ščedrov.) The points at which 'extra' '!'s are needed correspond to the points in the definition of 'Γ |- E' where the extra clause 'and Γ |- E is provable in Heyting Arithmetic' is required. (See, for example, Moschovakis [1980], Definition 1.9.) What, if anything, this means is not known.
In what follows, we will work with the two-sided sequent calculus for modal linear logic, DLL! (‘D’ for ‘double’); the system is presented in Appendix B. DLL! makes for a smoother transition to Gentzen’s sequent calculus for intuitionistic logic LJ, and from Gentzen’s Natural Deduction system for intuitionistic logic NJ (actually, just the propositional subsystems thereof).

**Theorem 3.3.1:** (Girard (1987))

Let \( A \) be any formula of LJ. Given a cut-free proof \( \pi \) of \( \vdash A^o \) in DLL!, we can construct a (unique) cut-free proof \( J(\pi) \) of \( \vdash A \) in LJ.

**Proof:**

Let \( \pi \) be a cut-free proof of \( \vdash A^o \) in DLL!.

Now \( A^o \) is a formula in the \{ \( \mathcal{O}, \&, \oplus, \neg, ! \} \) fragment of the language \( L_0! \) (in which \( \neg \), and \( (\neg)^* \), are primitive). Hence \( \pi \) involves only rules from the \{ \( \mathcal{O}, \& , \oplus, \neg, ! \} \) subsystem of DLL!, i.e. the identity axiom scheme, (EXCH), (CUT), the axiom scheme (\( \mathcal{O}L \)) for \( \mathcal{O} \) and logical rules for the connectives \&, \oplus, \neg \, and \!.

By examining these rules, we can convince ourselves that all sequents occurring in the proof \( \pi \) contain only one formula on their right-hand sides. Now we obtain \( J(\pi) \) from \( \pi \) as follows: (i) erase all occurrences of \(!\); (ii) replace each occurrence of \( \mathcal{O} \) with \( \mathcal{F} \); and (iii) replace each occurrence of \&, \oplus, \neg \, with \( \land, \lor, \Rightarrow \) respectively. Then \( J(\pi) \) is a cut-free proof of \( \vdash A \) in LJ. ■
Theorem 3.3.2: (Girard [1981])

Let \( A \) be any formula of \( L_\mathcal{I} \).

Given a deduction \( d \) of \([\Delta] A\) in \( NJ\),

i.e. \( d \) is a deduction of \( A \) from a set of assumptions \( \Delta \),

we can construct a (unique) proof \((d)^o\) of \( !\Delta^o \vdash A^o \)

in\( DLI! \).

(Where if \( \Delta = \emptyset \) \( \{c_1, \ldots, c_n\} \), then \( !\Delta^o \vdash A^o \) means \( !(c_1^o), \ldots, !(c_n^o) \vdash A \).

Proof:

The proof \((d)^o\) is defined by induction on \( d \).

(i) \( d \) is the deduction of \([A] A\) consisting of the assumption \( A \).

Then \((d)^o\) is

\[
\frac{A^o \vdash A^o}{!A^o \vdash A^o} \quad (L.D!)
\]

(ii) \( d \) is a deduction of \([\Delta_1, \Delta_2] A_1 \land A_2\) obtained from a deduction \( d_1 \) of \([\Delta_1] A_1\) and a deduction \( d_2 \) of \([\Delta_2] A_2\) by the rule of \( \land \) introduction.

Then \((d)^o\) is as follows:

\[
\frac{\vdots}{!\Delta_1^o \vdash A_1^o} \quad \text{Several (LTh!)}
\]

\[
\frac{\vdots}{!\Delta_2^o \vdash A_2^o} \quad \text{Several (LTh!)}
\]

\[
\frac{!\Delta_1^o, !\Delta_2^o \vdash A_1^o \land A_2^o \quad (R\&)}{!\Delta_1^o \land !\Delta_2^o \vdash A_1^o \land A_2^o \land}
\]

And \( (A_1 \land A_2)^o \) = \( A_1^o \land A_2^o \).
(iii) $d$ is a deduction of $[\Delta] A_i$ obtained from a deduction $d_1$ of $[\Delta] A_i \land A_2$ by the first $\land$ elimination rule. Then $(d)^o$ is as follows:

\[
\begin{align*}
\vdots \\
\neg \Delta^o \vdash A_i^o \land A_2^o & \quad \text{ (L\&I)} \\
A_i^o, A_2^o \vdash A_i^o & \quad \text{ (cut)} \\
\neg \Delta^o \vdash A_i^o
\end{align*}
\]

(iv) The case for the second $\land$ elimination rule is symmetric to (iii).

(v) $d$ is a deduction of $[\Delta] A_i \lor A_2$ obtained from a deduction $d_1$ of $[\Delta] A_i$ by the first $\lor$ introduction rule. Then $(d)^o$ is as follows:

\[
\begin{align*}
\vdots \\
\neg \Delta^o \vdash A_i^o & \quad \text{ (R!)} \\
\neg \Delta^o \vdash \neg (A_i^o) & \quad \text{ (R\oplus I)} \\
\neg \Delta^o \vdash \neg (A_i^o) \oplus \neg (A_2^o)
\end{align*}
\]

And $(A_i \lor A_2)^o = \neg (A_i^o) \oplus \neg (A_2^o)$.

(vi) The case for the second $\lor$ introduction rule is symmetric to (v).

(vii) $d$ is a deduction of $[\Delta, \Gamma, \Gamma_2] A$ obtained from deductions $e$ of $[\Delta] B_1 \lor B_2$, $d_1$ of $[\Gamma, \Xi_1] A$, and $d_2$ of $[\Gamma_2, \Xi_2] A$, where $\Xi_1$ and $\Xi_2$ consist of a number of repetitions of $B_1$ and $B_2$ respectively, (possibly zero) by $\lor$ elimination.
Then \((d)^o\) is as follows:

\[
\begin{array}{c}
\frac{\vdash \Delta \vdash !C_1 \oplus !C_2}{\vdash \Delta \vdash !C_1 \oplus !C_2} \quad \text{(L-Th)}
\end{array}
\]

[i] indicates several \((\text{LC}!)\) or one \((\text{LTh})\);

[ii] indicates several \((\text{LTh})\).

(viii) \(d\) is a deduction of \([\Delta] A \Rightarrow B\) obtained from a deduction \(d_1\) of \([\Delta, \Gamma] B\), where \(\Gamma\) consists of a number of repetition of \(A\) (possibly zero) by \(\Rightarrow\) introduction. Then \((d)^o\) is as follows:

\[
\begin{array}{c}
\frac{\vdash \Delta \vdash !A \vdash B}{\vdash \Delta \vdash !A \vdash B} \quad \text{(R-o)}
\end{array}
\]

And \((A \Rightarrow B)^o = \vdash !A \Rightarrow B^o\)

(ix) \(d\) is a deduction of \([\Delta, \Gamma] B\) obtained from a deduction \(d_1\) of \([\Delta] A\) and a deduction \(d_2\) of \([\Gamma] A \Rightarrow B\), by \(\Rightarrow\) elimination.

Then \((d)^o\) is as follows:
(x) \( d \) is a deduction of \([\Delta]A\) obtained from a deduction \(d_1\) of \([\Delta]f\) by the \(f\) rule. Recall that \((f)^* = \emptyset\).

Then \((d)^*\) is as follows:

\[
\begin{align*}
\frac{! \Delta^o \vdash \emptyset \quad \emptyset \vdash B^o}{! \Delta^o \vdash B^o}
\end{align*}
\]

We dont have to deal with \(\neg\) introduction or \(\neg\) elimination as we can take \(\neg A \iff A \Rightarrow f\).
4. What does it all mean?

4.0 A steady-state economy

In this section we give an intuitive, informal account of the connectives of propositional linear logic. Our discussion draws on Martí-Oliet and Meseguer [1989], Lafont [1988a], Appendix A, and Girard [1989], Section II.

We think of formulae as denoting states and interpret the linear implication connective \( \rightarrow \) as expressing the possibility of transition between two given states. So

\[ A \rightarrow B \]

is read as meaning that states of type \( B \) are accessible from states of type \( A \).

Consider the following example:

\[ A := \text{having a one dollar coin;} \]
\[ B := \text{getting a felafel;} \]
\[ C := \text{getting a doughnut;} \]
\[ D := \text{having five twenty cent coins}. \]

Our system is a little economy: transitions are monetary transactions. We rule out barter, i.e., exchange of one sort of non-monetary good for another. All resources or commodities in our economy are of equal value, i.e., worth one dollar. Our guiding principle is that this is a 'steady-state' economy: the total value of resources after a transaction is the same as it was before that transaction. So no value is lost and
no value is accrued.

The basic permissible transactions in our economy are given by the following diagram.

![Diagram](image)

More formally, we need a language of propositional linear logic generated by four propositional letters $A, B, C$ and $D$, and a theory with six non-logic axioms (given by the arrows other than the reflexive ones). With respect to the two-sided sequent calculus DLL, the non-logical axioms are $A \vdash B$, $A \vdash C$ etc. For more on linear theories and their connections with petri nets, all cast in a category-theoretic framework, see Martí-Oliet and Meseguer [1989].

Returning to our informal discussion, we have

- $A \rightarrow B :=$ with a one dollar coin we can get a falafel;
- $D \rightarrow A :=$ we can exchange five twenty cent coins for a one dollar coin;
- $A \rightarrow A :=$ we start with a dollar coin and end up with a dollar coin (maybe a different coin).
Instances of $A$, $B$, $C$ and $D$ correspond to credits on our ledger; their negations are debits. First, the multiplicative falsum:

$$\bot := \text{being obliged to pay}.$$ 

Then we have:

$$B^\bot \equiv_{\bot} B \rightarrow \bot := \text{We have a felafel and then are obliged to pay, i.e. We eat or otherwise dispose of the felafel.}$$

The dual diagram:

\[ \text{The dual diagram graph with nodes labeled } A^\bot, B^\bot, C^\bot, \text{ and } D^\bot. \]

Represents permissible debt transactions. For example,

$$B^\bot \rightarrow A^\bot := \text{if we eat a felafel then we can fulfil our obligation to pay by handing over a dollar coin;}$$

$$A^\bot \rightarrow D^\bot := \text{a debt of a one dollar coin can be replaced by a debt of five twenty cent coins.}$$
The multiplicative conjunction is interpreted as an accumulation of resources. We have $A \rightarrow B$ and $A \rightarrow C$ but from these we cannot derive $A \rightarrow (B \& C)$.

$$A \rightarrow (B \& C) := \text{with a one dollar coin we can get two dollars worth of goods, one felafel and one doughnut.}$$

This would mean getting something for free. The best we can do is $(A \& A) \rightarrow (B \& C)$. Returning to our favourite formulae, characteristic of contraction and thinning respectively,

$$A \rightarrow (A \& A) := \text{producing a dollar coin out of thin air;}$$

$$(A \& B) \rightarrow A := \text{having a felafel vapourize before our eyes.}$$

But what about the additive conjunction? From $A \rightarrow B$ and $A \rightarrow C$ we can derive $A \rightarrow (B \& C)$. Girard sees \& as involving the superposition of states:

$$A \rightarrow (B \& C) := \text{having a one dollar coin, we can choose between getting a felafel and getting a doughnut; both possibilities are available — we get to choose which one is realized.}$$

Recall that \& is idempotent (while \& is not): choosing between a felafel and a felafel is no choice at all.
With \&, something external to the system (in this case, us) gets to make a choice. There is also a choice involved with \( \oplus \), but the choice is not ours to make. From \( A \rightarrow C \), we can derive \( A \rightarrow (C \oplus D) \).

\[
A \rightarrow (C \oplus D) := \text{starting with a one dollar coin, we will get either a doughnut or five twenty cent coins; the choice is made by some mechanism internal to the system.}
\]

In this system, both \( C \) and \( D \) are accessible from \( A \). We can also derive \( A \rightarrow (C \oplus (D \oplus D \oplus D)) \); in that case, the internal mechanism will always choose \( C \).

The choice involved with \( \oplus \) may seem a bit trivial: the 'internal mechanism' should always choose the disjunct from which the disjunction was inferred. Maybe it should, but sometimes it can't make the decision on that basis. Consider the following proof in LL (which we've seen before but is repeated here to make the point):

\[
\begin{align*}
\vdash & C^+, \neg C^+ \\
\vdash & C^+, \neg C^+ (\text{fs} \oplus) \\
\vdash & D^+, \neg D^+ \\
\vdash & D^+, C \oplus D (\text{snb} \oplus) \\
\vdash & C^+ \land D^+, C \oplus D (\&) \\
\vdash & C \oplus D \rightarrow C \oplus D (\%)
\end{align*}
\]

With regard to the '\( C \oplus D \)' occurring in the consequent of the final implication, backtracking through the proof to see which of the two \( \oplus \) rules were used will be of no help in trying to predict the decision of the 'internal mechanism'.
As one might guess from the above discussion, things are different when $\&$ or $\oplus$ occur in the antecedent of an implication, (or on the left hand side of a sequent). It seems as though the 'external/internal choice' characterization is reversed. From $A \rightarrow B$ and $D \rightarrow B$, we can derive $(A \oplus D) \rightarrow B$. Recalling that $(A \oplus D) \rightarrow B \equiv_{LL} (A \rightarrow B) \& (D \rightarrow B)$, the obvious reading is

$$(A \oplus D) \rightarrow B := \text{We can choose between spending a one dollar coin and spending five twenty cent coins when purchasing a falafel; both possibilities are available.}$$

Now consider again $(C \oplus D) \rightarrow (C \oplus D)$. If we choose $C$ in the antecedent then it (the 'internal mechanism') must choose $C$ in the consequent (since $D$ is not accessible from $C$). But if we choose $D$ in the antecedent, then it can take its pick. (There seems to be a need for a general policy about priority of choices when there is competition; something like 'We always choose first, of course!' might do it.) For 'dual' reasons, the choice between starting places in, say, $(A \& D) \rightarrow B$ is most naturally read as an internal choice.

Note, however, that the 'accumulation' reading of $\oplus$ works, regardless of which side of the $\rightarrow$ the $\oplus$ occurs. For example:

$$(A \oplus A \oplus D) \rightarrow (B \oplus C \oplus C) := \text{With two one dollar coins and five twenty cent coins we can get one falafel and two doughnuts.}$$
The multiplicative disjunction $\otimes$ is more problematic. As Girard ([1989], p.73) rightly says, '...the meaning of $\otimes$ is not that easy.' One way of thinking of it is in terms of $\rightarrow$, i.e. $A \otimes B = A^+ \rightarrow B$ by definition. This is certainly the best way of thinking of the 'excluded middle' axioms $A \otimes A^+$. The other way of thinking of $\otimes$ is as the dual of $\otimes$: instead of accumulation of resources we have accumulation of debts. For example:

$$(B^+ \otimes C^+) \rightarrow (A^+ \otimes A^+) \equiv \text{We can replace a debt of one felafel and one doughnut by a debt of two one dollar coins.}$$

This, however, gets us no closer to understanding what the $\otimes$ of positive formulae means. We are left with the feeling that $\otimes$ is by far the weirdest of the connectives of linear logic.

4.1 A licence to print money (!) and limited versions thereof.

When we adjoin the modal operators to linear logic, the steady-state economy is replaced by economic chaos:

$!A := \text{unlimited possibilities for duplicating one dollar coins.}$

The inference from $\vdash A, ?\Delta$ to $\vdash !A, ?\Delta$ using (!) is analogous to putting $A$ in a memory register so that it can be used ad nauseum. As Girard notes ([1989], p.84)
the four modal rules of LL! differ from the rules for the additives and multiplicatives in that they do not completely characterize their connectives. If we were to introduce a second pair of additives \&' and \Theta' with the same rules as for \& and \Theta then A\&'B and A\Theta'B will be provably equivalent to A\&B and A\Theta B respectively. The same holds for \Theta and \S, but it does not hold for ! and ?. To establish !A \equiv !'A we need proofs of both \vdash ?A', !'A and \vdash ?'A, !A. We can get both \vdash ?A', A and \vdash ?'A, A but then get stuck because we have the wrong sort of question mark. However, since we want ! and ? to be dual, then given the (!) rule, the 'inverse' rule (D?) is forced upon us; (D?) in conjunction with (CUT) is analogous to retrieving data from a memory register. Continuing the metaphor, the rules (th?) and (C?) are 'optional extras' which give us power to manipulate data in the memory register.

This power was crucial in the translation of an NJ derivation into a DLL! proof (Theorem 3.3.2).

A note of caution is in order: the modal operators introduced in Girard and Lafont [1987] and Lafont [1988b], in the context of intuitionistic linear logic, do differ from those under discussion here; see Lafont [1988b], pp.164-165 and Seely [1989], p.379.

The operators ! and ? of LL! are perhaps best thought of as infinite generalizations of \Theta and \S, similar to the sense in which the quantifiers \Lambda and \Sigma are infinite generalizations of \& and \Theta, but not quite.
In Girard [1987] we have the proposal:

\[ !A \approx \bigotimes_{\omega} (1 \& A) ; \]
\[ ?A \approx \bigotimes_{\omega} (\bot \oplus A) . \]

(Since we have trouble understanding what \(A \& B\) means, an infinitary \(\bigotimes\) is going to fair no better at the level of Comprehension.) Note that \(\vdash (1 \& A) \rightarrow A\) and \(\vdash A \rightarrow (\bot \oplus A)\) are provable in LL, but in general their converses are not. \(\vdash A \rightarrow (1 \& A)\) is provable in LL! when \(A\) is \(!B\), or more generally, when \(\vdash A \rightarrow \bot\) is provable. A story might be of some help. When we want to make use of \(!A\), we begin by considering the first in our string of \((1 \& A)\)'s. A choice has to be made (either by us or by it) between \(\bot\) and \(A\). The choice of \(\bot\) is 'default mode' — \(\bot\) contains no new information. If the choice is ours, then we choose as many copies of \(A\) as we need to get on with the proof. For example, if there is a cut with \(?A^\ast\) then the number of copies of \(A\) we choose may depend on the number of copies of \(A^\ast\) it has chosen from the list of \((\bot \oplus A^\ast)\)'s. If we are simulating an NJ derivation (as in theorem 3.3.2) in which \(A\) is an assumption formula which is used \(K\) times, then we'll keep on going through the \((1 \& A)\)'s until we've got \(K\) copies of \(A\). We have an unlimited number of possibilities for reproducing \(A\), in contrast to the scenario in which \(!A\) is \(\bigotimes_{\omega} A\), the latter gives us an unlimited number of actual copies of \(A\).
The story gives some informal content to Girard's slogan that 'intuitionistic logic is obtained from [non-modal] linear logic by a passage to the limit.' Now for the mathematical content.

**Definition 4.1.0:**
Let $A$ be any formula in the language $L$ of propositional linear logic. For each positive integer $n$, we define

$$!_n A \overset{\text{df}}{=} (1 \land A) \otimes \ldots \otimes (1 \land A) \quad (n \text{ times}),$$

$$?_n A \overset{\text{df}}{=} (1 \lor A) \& \ldots \& (1 \lor A) \quad (n \text{ times}).$$

**Lemma 4.1.1:**
The following are derived rules in $LL$:

$$\frac{\Gamma}{\Gamma \vdash ?_n A, \Delta} \quad (\text{ap. Th?})$$

$$\frac{\Gamma \vdash ?_m A, ?_n A, \Delta}{\Gamma \vdash ?_{n,m} A, \Delta} \quad (\text{ap. C?})$$

$$\frac{\Gamma \vdash A, \Delta}{\Gamma \vdash ?_n A, \Delta} \quad (\text{ap. D?})$$

$$\frac{\Gamma \vdash A, ?_n \Delta}{\Gamma \vdash !_{kn} A, ?_{kn} \Delta} \quad (\text{ap. !})$$

where $n, m, k \geq 1$, $?_n \Delta$ abbreviates $?_{n_1}, \ldots, ?_{n_p} C_p$ with $n_i \geq 1$, $i=1, \ldots, p$, and $?_{kn} \Delta$ abbreviates $?_{k_1}, \ldots, ?_{k_n} C_p$.

('ap.' is for approximation.)

**Proof:**
Almost immediate. (ap. C?) requires only the ($\&$) rule,
(ap. D?) uses the (SND $\oplus$) rule with $\bot$ as first disjunct
and (ap. Th?) requires an application of the rule ($\bot$) followed
by (FST $\oplus$) with $A$ as second disjunct. We give a derivation
for \((\text{ap}.!\)). Let \(\Pi_0\) be a proof in \(LL\) of \(\vdash A, ?^n B\) for some \(n \geq 1\). Assume that from \(\vdash A, ?^n B\) we can derive in \(LL\) the sequent \(\vdash !^{k}A, ?^{kn} B\), for \(k \geq 1\). Then

\[
\frac{\vdash 1}{\vdash 1, ?^n B} \quad \frac{\vdash A, ?^n B}{\vdash 1 \& A, ?^n B} \quad \frac{\vdash A, ?^n B}{\vdash A, ?^{kn} B} \quad \frac{\vdash 1 \& A, ?^n B}{\vdash !^{k}A, ?^{kn} B} \quad \frac{\vdash (1 \& A) \otimes !^{k}A, ?^n B, ?^{kn} B}{\vdash !^{k+1}A, ?^{n(k+1)} B}
\]

\([a]\) denotes \((\text{ap}.\text{Th})\)'s together with \((\text{ap}.\theta)\)'s. The base case of the induction is just the top left-hand part of the above. \(\blacksquare\)

Not surprisingly, the 'thinning-like' aspects of the rules for \(1\) and \(\otimes\) are exploited in giving an approximation of the modal thinning rule. In a less direct way, the implicit contraction involved in the rule \((\&)\) is used to get the nice polynomial property in \((\text{ap}.!\)).

**Lemma 4.1.2:**

The following are derived rules in \(LL\):

1. For any \(n < m\),

\[
\frac{\vdash 1}{\vdash ?^m A, \Delta} \quad \frac{\vdash !^m A, \Delta}{\vdash !^n A, \Delta} \quad (\text{ap}.?! \uparrow) \quad (\text{ap}.! \downarrow)
\]

**Proof:**

\((\text{ap}.?! \uparrow)\) just involves multiple applications of \((\text{ap}.\text{Th})\) followed by \((\&)\)'s. \((\text{ap}.! \downarrow)\) uses \((\text{ap}.?! \uparrow)\) together with \((\text{Cut})\):
\[ \vdash \forall A, !n A \quad \text{(cut)} \]

Suppose that we did translate \(!A\) as a straight infinitary \(\otimes\), and hence had approximants

\[ !^n A \equiv A \otimes \ldots \otimes A \quad \text{(n times)}; \]
\[ ?^n A \equiv A \& \ldots \& A \quad \text{(n times)}. \]

Then the approximation of the modal thinning rule, (op. Th\(?\)) is not sound with respect to phase space semantics, and hence the same fate befalls the analogous increase and decrease rules (op. ?\(\uparrow\)) and (op. \(!\downarrow\)), (which of course means these rules aren't derivable in \(\text{LL}\)).

To see that we can have a fact \(G\) fail to be verified but have \(G \otimes G\) verified, consider the phase space \(P = \langle \mathbb{Z}/5\mathbb{Z}, +, 0, \bot_p \rangle\) where \(\bot_p = \{1, 2\}\) and \(\top = \{0\}\). Let

\[ G = \{4\}^+ = \{ n \in \mathbb{Z} \mid n+4 \not\equiv 1, 2 \pmod{5}\} = \{2, 3\}. \]

So the identity \(0 \not\in G\). But \(GG = \{0, 1, 4\}\), hence \(0 \in G \otimes G = (GG)^\downarrow\). In fact, \(G \otimes G = \top\).

To put the complaint more directly, the would-be approximation (op. Th\(?\)) just is full-blown thinning:

\[ \vdash \Delta \]

\[ \vdash \Delta \]

\[ \vdash A, \Delta \]
And now to return to the task at hand.

**Theorem 4.1.3: Approximation Theorem (Girard [1987])**

If $\Pi$ is a cut-free proof in $LL$ of $\vdash A_1, \ldots, A_p$ and, for each $i = 1, \ldots, p$, we assign to each occurrence of $!$ in $A_i$ a positive integer (called the index of that occurrence of $!$), then, for each $i = 1, \ldots, p$, and each occurrence of $?$ in $A_i$, we can specify an index (positive integer) in such a way that if $A^*_i$ is the result of replacing each occurrence of $!$ in $A_i$ by $!k$, where $k$ is the index of that occurrence of $!$, and replacing each occurrence of $?$ in $A_i$ by $?m$, where $m$ is the index specified for that occurrence of $?$, then there is a cut-free proof $(\Pi)^*$ in $LL$ of $\vdash A^*_1, \ldots, A^*_p$.

**Proof:**

By induction on the cut-free proof $\Pi$.

(i) $\Pi$ is an instance of the axiom scheme $\vdash A, A^*$: the identity axiom scheme is restricted to literals, so there is nothing to do; similarly if $\Pi$ is the axiom $\vdash \mathbb{I}$.

(ii) $\Pi$ is obtained from $\Pi_0$ by the either the rule (EXCH) or the rule ($\bot$): obvious.

(iii) $\Pi$ is an instance of the axiom scheme $\vdash T, A_1, \ldots, A_n$: the $!$ indices for each $A_i$ place no restriction on the $?$ indices; we can set them equal to 1.
(iv) \( \Pi \) is obtained from \( \Pi_0 \) by the rule \((\text{FST} \Theta)\):
\[
\frac{\Gamma \vdash A, \Delta}{\vdash A \Theta B, \Delta} \quad (\text{FST} \Theta)
\]
Assign an index to each occurrence of \( ! \) in \( A \Theta B, \Delta \); this induces an assignment of \( ! \) indices in \( A, \Delta \). The induction hypothesis gives us a proof \((\Pi_0)'^*\) of \( \vdash A'^*, \Delta'^* \). Set all \( ? \) indices in \( B \) to 1 to give \( B'^* \). Then \((\Pi)^'*\) is obtained from \((\Pi_0)^'*\) by applying \((\text{FST} \Theta)\) with respect to \( B'^* \).
Likewise for the rule \((\text{SND} \Theta)\).

(v) \( \Pi \) is either obtained from \( \Pi_0 \) and \( \Pi_1 \) by \((\Theta)\), or else obtained from \( \Pi_2 \) by \((\theta)\): the occurrences of \( ! \) and \( ? \) in the conclusion sequent are exactly those of the premise sequents (or sequent) so the induction hypothesis gives us all we need.

(vi) \( \Pi \) is obtained from \( \Pi_0 \) by the rule \((\text{Th}?)\):
\[
\frac{\Gamma \vdash \Delta}{\vdash ?A, \Delta} \quad (\text{Th}?)
\]
Assign an index to each occurrence of \( ! \) in \( ?A, \Delta \), inducing an assignment of \( ! \) indices in \( \Delta \). By the induction hypothesis, we have a proof \((\Pi_0)^'*\) of \( \vdash \Delta'^* \). Set all \( ? \) indices in \( A \) to 1 to give \( A'^* \). Then apply the rule \((\text{ap. Th}?)\) to \((\Pi_0)^'*\).

(vii) \( \Pi \) is obtained from \( \Pi_0 \) by the rule \((\text{D}?)\):
\[
\frac{\Gamma \vdash A, \Delta}{\vdash ?A, \Delta} \quad (\text{D}?)
\]
As in case (vi), except that the induction hypothesis gives us a proof \((\Pi_0)^'*\) of \( \vdash A'^*, \Delta'^* \). Apply the rule \((\text{ap. D}?)\) to \((\Pi_0)^'*\) to give a proof \((\Pi)^'*\) of \( \vdash ?, A'^*, \Delta'^* \).
(viii) \( \Pi \) is obtained from \( \Pi_0 \) by the rule (I):

\[
\frac{\Gamma \vdash A, ?\Delta}{\Gamma \vdash !A, ?\Delta} \quad (I)
\]

Assign an index to each occurrence of \( ! \) in \( ?A, \Delta \); suppose the index of the first \( ! \) in \( !A \) is \( k \). The induction hypothesis gives a proof \( (\Pi_0)^* \) of \( \vdash A^*, ?\Delta^* \). Now apply (ap. !) to get a proof \( (\Pi)^* \) of \( \vdash !kA^*, ?k\Delta^* \).

The remaining two cases, the modal contraction rule (C2) and the rule (\&), pose greater difficulties precisely because of their contractive nature.

(ix) \( \Pi \) is obtained from \( \Pi_0 \) by the rule (C2):

\[
\frac{\vdash ?A, ?A, \Delta}{\vdash ?A, \Delta} \quad (C2)
\]

Assign an index to each occurrence of \( ! \) in \( ?A, \Delta \). This induces an assignment of \( ! \) indices in \( ?A, ?A, \Delta \); in particular, the corresponding \( ! \) indices in the two occurrences of \( ?A \) are identical. The induction hypothesis gives us a proof \( (\Pi_0)^* \) of \( \vdash ?nA', ?mA'', \Delta^* \) where \( A' \) and \( A'' \) are both approximants of \( A \), but may differ with respect to \( ? \) indices. If \( A' = A'' \) then apply the rule (ap. C2) to get a proof \( (\Pi)^* \) of \( \vdash ?n+mA', \Delta^* \). Otherwise, there are \( q \geq 1 \) occurrences of \( ? \) in \( A \) such that if \( i \) and \( j \) are the indices specified for that \( ? \) in \( A' \) and \( A'' \) respectively, then \( i \neq j \).

The following procedure is required.

Step one. Find the largest subformula \( ?D \) of \( A \) such that \( ?iD' \) and \( ?jD'' \) are the corresponding subformulae of \( A' \) and \( A'' \) respectively, with \( D' \) and \( D'' \) both approximants of \( D \), and \( i \neq j \).

If there are two or more occurrences of a subformula \( ?D \) of \( A \) with this property, or else there are two or more distinct subformulae of \( A \) with this property and all of maximal size,
then take the one that is in the scope of the least number of connectives; if there is still a tie, take the leftmost.
Assume $i < j$ (when $i > j$, the procedure is symmetric).

Step two. 'Back-track' up each branch of the proof (T10)*, making a note of those occurrences of $?i D'$ which are subformulæ of subformulæ of $A'$ and are direct ancestors of the occurrence of the $?i D'$ in $A'$ under investigation. Call these the lineage of the $?i D'$.

Step three. On each branch of (T10)* which involves the lineage of the $?i D'$, find the lowest sequent in which one of the lineage of the $?i D'$ occurs as a formula, i.e. the sequent which is the premiss of a rule in virtue of which one of the lineage of the $?i D'$ first becomes a proper subformula of some formula. Call such a sequent free for $?i D'$. On each branch of (T10)* with a sequent of the form $1 \Gamma', $?j D', \Sigma$ free for $?j D'$, apply the derived rule (ap.?1) to obtain a sequent of the form $1 \Gamma, $?j D', \Sigma$; then return to the proof (T10)* and replace all lower occurrences of $?i D'$ which are in the lineage of the $?j D'$ by $?j D'$. A branch of (T10)* which involves the lineage of the $?i D'$ will only fail to have a sequent free for $?i D'$ if the earliest member of the lineage of the $?i D'$ on that branch is either a subformula of the 'new' disjunct introduced by an application of one of the $\Theta$ rules or else is a proper subformula of a formula occurring in an instance of the axiom scheme for $T$.

Now duplicate that application of a $\Theta$ rule (that instance of the $T$ axiom scheme) except for replacing that (those) occurrence(s) of $?D'$ which are in the lineage of the $?i D'$ by $?j D'$, and do the same for all lower occurrences of $?i D'$ which are in the lineage of the $?i D'$. 
The result of these modifications to the proof \((\Pi_6)^*\) will be a proof \(\mu\) in LL of \(\vdash ?_n A'', ?mA'', \Delta''\) where \(A''\) is the same as \(A'\) except that the occurrence of \(?_i D'\) has been transformed into \(?_j D'\). We claim that \(A''\) and \(A''\) have exactly \(q-1\) discrepant pairs of \(?\) indices and hence, by the obvious induction, we will have after \(q\) iterations of the three-step procedure a proof \(\mu_q\) in LL of 
\[\vdash ?_n A^*, ?mA^*, \Delta^*.\]
Our claim is based on the fact that by always working on the outermost discrepant \(?\) indices and by only increasing indices in the lineage of the occurrence of the subformula under investigation, we never create a discrepancy of \(?\) indices and once a discrepancy has been fixed, we never have to attend to it again.

\[(x)\] \(\Pi\) is obtained from \(\Pi_0\) and \(\Pi_i\) by the rule (\(\&\)):
\[
\begin{array}{c}
\vdash A, C, ..., C_r \hline
\vdash B, C, ..., C_r \quad (\&)
\end{array}
\]
\[\vdash A \& B, C, ..., C_r.\]

This time we have proofs \((\Pi_6)^*\) and \((\Pi_i)^*\) in LL of 
\[\vdash A^*, C^*, ..., C^r \quad \text{and} \quad \vdash B^*, C^*, ..., C^r.\]
respectively, where for \(i=1, ..., r\), \(C_i\) and \(C_i^*\) are both approximants of \(C_i\) but may differ with respect to \(?\) indices. The procedure is analogous to that given in case \((ix)\). For each discrepant pair of \(?\) indices, we have to either make suitable modifications to \(?\) indices in \((\Pi_6)^*\), or else do likewise for \((\Pi_i)^*\). The result gives proofs \(\mu_0^*\) and \(\mu_i^*\) in LL of 
\[\vdash A^*, C^*, ..., C^r \quad \text{and} \quad \vdash B, C^*, ..., C^r.\]
respectively; then obtain the proof \((\Pi)^*\) of 
\[\vdash A^* \& B^*, C^*, ..., C^r\]
by (\(\&\)).

Unfortunately, the effort required to fill in the missing details in Girard's proof [1987], p. 93, turned out to be much greater than the author first anticipated.
An obvious consequence of the Approximation Theorem is that given any theorem of propositional intuitionistic logic (or provable sequent of propositional LJ) we can produce approximants of its translate which are provable in LL. However, we don’t have approximants for the translate in LL! of each proof in propositional LJ (going via Gentzen’s translation of LJ proofs into NJ derivations, then applying Theorem 3.3.2); the hypothesis of the Approximation Theorem that the proof in LL! be cut-free is crucial since we need the subformula property.

The polynomial structure underlying the ‘exponential’ (modal) connectives ! and ? is of particular note.

**Corollary 4.1.4:**
If \( \Pi \) is a cut-free proof in LL! of \( \vdash \Delta \)
and \( x_1, \ldots, x_s \) are distinct variables associated with the \( s \) occurrences of ! in \( \Delta \) in, say, left to right order, then for each occurrence of ? in \( \Delta \) there is a polynomial function \( p(x_1, \ldots, x_s) \) such that, given assignments \( k_1, \ldots, k_s \) of positive integers to the variables \( x_1, \ldots, x_s \), if \( \Delta^* \) is obtained from \( \Delta \) by replacing the \( i \)th occurrence of ! in \( \Delta \) by !\( k_i \), for \( i = 1, \ldots, s \), and replacing each occurrence of ? in \( \Delta \) by ?\( p(k) \), where \( p \) is the polynomial for that ? and \( p(k) = p(k_1, \ldots, k_s) \),
then there is a cut-free proof \( (\Pi)^* \) in LL of \( \vdash \Delta^* \).
Proof:
By induction on the cut-free proof $\Pi$. In cases except (C?) and (G), the polynomial in question falls out of the proof of the Approximation Theorem. Suppose we have a proof in $LL$ of $\Gamma \vdash ?_f A', ?_g A'', \Delta^*$, i.e. the polynomials for the displayed $?$ in $\Gamma \vdash ? A, ? A, \Delta$ are $f$ and $g$ respectively, and $A'$ and $A''$ are polynomial approximants of $A$. As in case (ix) of the proof above, we start with the outermost and leftmost occurrence of $?$ in $A$ which gives rise to a discrepancy between $A'$ and $A''$. Suppose $p$ is the polynomial for that occurrence of $?$ in $A'$ and $q$ is the polynomial for that occurrence of $?$ in $A''$ and $p \neq q$. To repair the discrepancy, we replace both $p$ and $q$ by the polynomial $\max\{p,q\} = p + (q - p)$, and do the same for both the ancestors of that $?$ in $A'$ and the ancestors of that $?$ in $A''$. A similar manoeuvre works for the rule (G).

In recent work, Girard, Ščedrov and Scott [1990] have developed a system called Bounded Linear Logic (BLL) with modalities of the form $!x:p A$ where $p$ is a certain type of polynomial not containing the variable $x$. (They use a two-sided sequent calculus and ignore '$?'.)

BLL also has full, impredicative second-order quantification so types of natural numbers are definable, but the bounded $!$'s give rise to distinct levels of natural numbers. Their main result is that there is a one-one correspondence between the class $P$ of polynomial-time computable functions and a certain class of closed terms in BLL.
5. **Coherent space semantics**

The phase space semantics provided, in a Tarskian manner, an algebraic interpretation of formulae of linear logic. An alternative approach to semantics, that of Heyting, involves the modelling of proofs. In this tradition, proofs are associated with functions or relations on some kind of constructive space. Dana Scott's theory of domains (eq. Scott [1982]) is prominent in this tradition. Girard reworks Scott's theory to create a new kind of domain which he calls coherent spaces. It is within this framework that we get to see how linear implication emerges from a decomposition of intuitionistic implication. In this section, we draw on material from Girard, Lafont and Taylor [1989] and Lafont [1988a] in order to fill in the background of Chapter 3 of Girard [1987].

5.0 **Coherent spaces and webs**

**Definition 5.0.1:**

A coherent space $X$ is a family of sets satisfying the following conditions:

(i) if $a \in X$ and $b \in a$ then $b \in X$;

(ii) if $S \subseteq X$ and for all $a, b \in S$, $a \cup b \in X$ then $U \subseteq X$.

If $a \in X$ then $a$ is called an object of $X$.

Considering a coherent space $X$ as partially ordered by inclusion, (i) says that $X$ is closed downward.
As a consequence, $\varnothing \in X$; $\varnothing$ is the undefined object of $X$. Furthermore, if $a, b \in X$ then $a \cap b \in X$ by (ii). Indeed, if $S$ is any non-empty subfamily of $X$, then $\cap S \in X$. When $a, b \in X$, $a \cup b$ will often fail to be an object of $X$. For most coherent spaces $X$, $\cap \varnothing = UX \notin X$. Call a subfamily $S \subseteq X$ (binarily) linked iff for all $a, b \in S$, $a \cup b \in X$. Combining conditions (ii) and (iii), we have $US \in X$ iff $S$ is linked. Condition (iii) is known as binary completeness.

Recall that a family of sets $S$ is (upwards) directed iff for all $a, b \in S$ there exists $c \in S$ such that $a \cup b \subseteq c$. For a coherent space $X$, if $S \subseteq X$ is directed then $S$ is linked, but not conversely. If $S$ is directed, we have $U^+S \in X$. For any object $a \in X$, the collection of all finite subsets of $a$ is directed, and

$$a = U^+a = \{a_0 \subseteq a \text{ and } a_0 \text{ finite}$$. This means that coherent spaces are, in the terminology of universal algebra, algebraic conditionally complete inf semi-lattices with binary completeness with respect to sups. (See, for example, Burris and Sankappanavar [1981].) In what follows, we'll be particularly interested in the directed family

$$X_{\text{fin}} = \{a \in X \mid a \text{ finite}\}$

where $X$ is a coherent space.
Consider a few examples of (very simple) coherent spaces.

\( \text{Emp} \triangleq \{ \emptyset \} ; \)
\( \text{Sgl} \triangleq \{ \emptyset, \{ \emptyset \} \} \) where \( \emptyset \) is some arbitrarily chosen, fixed element;
\( \text{Bool} \triangleq \{ \emptyset, \{ \text{true} \}, \{ \text{false} \} \} ; \)
\( \text{Int} \triangleq \{ \emptyset \} \cup \{ \{ n \} | n \in \mathbb{N} \} . \)

**Definition 5.0.2:**
Let \( X \) be a coherent space. The web of \( X \), denoted \( \mathcal{W}(X) \), is a reflexive, unoriented graph defined as follows:

(i) the domain of the graph,
\[ |X| \triangleq \cup X = \{ x | \{ x \} \in X \} ; \]

(ii) the edge relation, coherence modulo \( X \), is given by
\[ x \sim x' [\mod X] \text{ iff } \{ x, x' \} \in X . \]

The second identity in (ii) is due to the down closure of \( X \). It is important to note that the coherence mod \( X \) relation is not in general transitive: when \( x \sim x' [\mod X] \) and \( x' \sim x'' [\mod X] \), we will have \( x \sim x'' [\mod X] \) only if \( \{ x, x', x'' \} \in X \).

We can now see a coherent space \( X \) as a subset of \( P(1 \times 1) \). Observe that, for any subset \( a \subseteq 1 \times 1 \),
\[ a \in X \text{ iff } \text{for all finite } a_0 \subseteq a, \ a_0 \in X . \]

The crucial point is that not all finite subsets of \( 1 \times 1 \) are objects of \( X \), except in extreme cases.
Proposition 5.0.3: (Girard [1987])

The map \( W \) from the class of coherent spaces to the class of reflexive, unoriented graphs given by \( X \mapsto W(X) \) is a bijection. Moreover, the coherent space may be recovered from its web by means of the equivalence:

\[
a \in X \iff a \subseteq |X| \text{ and for all } x, x' \in a \left( x \leq x' \mod X \right).
\]

Proof:

We first establish the equivalence. Let \( a \in X \). Then \( a \subseteq |X| \), and for any \( x, x' \in a \), we have \( \{x, x'\} \subseteq a \) hence \( \{x, x'\} \in X \) by down closure. Conversely, assume \( a \subseteq |X| \) and for all \( x, x' \in a \), \( x \leq x' \mod X \). Then \( \{x, x'\} \in X \) for all \( x, x' \in a \). Hence \( \bigcup \{x \mid x \in a \} = a \in X \) by binary completeness.

In virtue of the equivalence, we can say that an object of \( X \) is a \underline{coherent subset} of \( |X| \); as such, it is the underlying set of a \underline{complete subgraph} of \( W(X) \). Translating the conditions in the definition of coherent spaces into graph-theoretic terms, down closure says that any subgraph of a complete subgraph is complete, and binary completeness says that if \( S \) is a family of complete subgraphs, any two of which are \underline{linked} in the sense that between any node of one and every node of the other, there is an edge, then the union of \( S \) is a complete subgraph. Both of these properties hold of reflexive, unoriented graphs. That gets surjectivity; injectivity is obvious.
We will informally assume, but not stipulate definitively, that webs are countable (which means coherent spaces may have the power of the continuum). If one is particularly concerned with constructivity, then one should assume that the coherent spaces in question have webs whose coherence relation is recursive; we will frequently have to 'exclude middles' and assume that for all \( x, x' \in |X| \), if \( x \neq x' \) then either \( x \subseteq x' \mod x \) or not \( x \subseteq x' \mod x \); i.e. either \( \{x, x'\} \in X \) or \( \{x, x'\} \notin X \). Which means membership and identity had better be recursive. In analogy with Scott's theory of domains, we can think of the \( x \in |X| \) as finite pieces of data.

In what follows, we will work mainly with webs; in particular, we define coherent spaces by specifying the domain and coherence relation of the web.

If \( X \) and \( Y \) are coherent spaces then a graph homomorphism from \( W(X) \) into \( W(Y) \) is a function \( f: |X| \rightarrow |Y| \) such that

\[ x \subseteq x' \mod x \iff f(x) \subseteq f(x') \mod y \]

for all \( x, x' \in X \). Such a function induces a (unique) function \( \bar{f}: X \rightarrow Y \) defined, for each \( a \in X \), by

\[ \bar{f}(a) = \{ f(x) \mid x \in a' \} \]

Observation 5.0.4:

If \( X \) and \( Y \) are coherent spaces and \( f: |X| \rightarrow |Y| \) is a graph homomorphism from \( W(X) \) into \( W(Y) \) then the function \( \bar{f}: X \rightarrow Y \) has the following properties:
(I) monotonicity: if \( a, a' \in X \) and \( a \in a' \) then \( f(a) \subseteq f(a') \);  
(II) preservation of unions (those that there are): 
\[ f(U_{a \in S} f(a)) = \bigcup_{x \in S} f(x) \; | \; a \in S \];  
(III) preservation of non-empty intersections: 
\[ f(\cap S) \subseteq \bigcap_{x \in S} f(x) \; | \; S \) non-empty \];  
(IV) preservation of singletons: for all \( x \in \text{X} \), \( f(\{x\}) = \{f(x)\} \);  
(V) strong preservation of undefined objects: \( f(\bot) = \emptyset \) iff \( a = \emptyset \).  

(Verification involves mindlessly writing out definitions.)

In short, graph homomorphisms induce functions on coherent spaces which preserve every bit of structure there is. Let us say that a function \( g : X \rightarrow Y \) is a **coherent space homomorphism** iff \( g \) satisfies conditions (I)-(V) above. Such a \( g \) uniquely determines a graph homomorphism \( \hat{g} : \text{X} \rightarrow \text{Y} \) defined by the equivalence

\[ \hat{g}(x) = y \iff g(\{x\}) = \{y\} \].

Hence we have an isomorphism from the category of coherent spaces with coherent space homomorphisms onto the category of reflexive, unoriented graphs with graph homomorphisms given by

\[ X \mapsto W(X) \; | \; g : X \rightarrow Y \mapsto \hat{g} : \text{X} \rightarrow \text{Y} \].

We say that coherent spaces \( X \) and \( Y \) are **canonically isomorphic**, written \( X \cong Y \), iff there is a graph isomorphism \( f : \text{X} \rightarrow \text{Y} \) defined uniformly for all arguments, by which we mean natural in the category-
theoretic sense.

In what follows, the classes of functions we need to characterize the function spaces \( X \Rightarrow Y \) and \( X \rightarrow Y \) will preserve rather less structure than coherent space homomorphisms (and hence be more interesting). The list (I)–(V) of properties is useful in providing a yardstick (although note that the list contains redundancies: e.g. (II) implies (I)) as well as clarifying what we take Girard to mean by 'canonical isomorphism' - Girard [1987], pp. 48-49.

Recall our examples of coherence spaces. The domain of their webs are as follows:

\[
\begin{align*}
|\text{Sgl}| &= \{ \ast \} \\
|\text{Bool}| &= \{ \ast, \top, \bot \} \\
|\text{Int}| &= \mathbb{N}
\end{align*}
\]

In each case, the coherence relation is such that for all \( x, x' \in |X| \), \( x \leq x' \mod X \) iff \( x = x' \), hence the webs are discrete graphs. Such coherence spaces are called flat. Up to isomorphism, there is only one flat coherent space of a given cardinality. For the case of cardinality zero we have \( \text{Emp} \), where \( |\text{Emp}| = \emptyset \).

At the other extreme are coherent spaces \( X \) such that for all \( x, x' \in |X| \), \( x \leq x' \mod X \). Here, \( W(X) \) is a complete graph. Observe that \( UX = |X| \in X \), so by down closure, \( X = P(|X|) \). Hence \( X \) is a complete boolean algebra. We propose the name \text{round} for such
Coherent spaces. Observe that Emp and Sgl are the only coherent spaces that are both flat and round.

A Word on notation and terminology is in order. In Lafont [1988a], the construction is reversed: Lafont first defines what we call webs and then defines what we call coherent spaces. A straight translation of Lafont [1988a] into Girard [1987] is as follows:

- coherent space, \( A \mapsto |X| \), domain of \( W(X) \);
- coherent subsets of \( A \), \( \text{Con}_A \mapsto X \), coherent space.

Coherent spaces in the sense of Girard [1987] are relatives of domains in the sense of Scott. Following the notation of Scott [1982], we give a 'wobbly' correspondence with Girard [1987]:

given an information system \( A = \langle D_A, \Delta_A, \text{Con}_A, \preceq_A \rangle \),

- the domain determined by \( A \), \( |A| \mapsto X \), coherent space;
- the data objects of \( A \), \( D_A \mapsto |X| \), domain of \( W(X) \);
- the undefined element \( \Delta_A \in D_A \), \( \Delta_A \mapsto \emptyset \in X \);
- a family of finite subsets of \( D_A \), \( \text{Con}_A \mapsto X_{\text{fin}} \).

Girard drops Scott's entailment relation \( \preceq_A \subseteq \text{Con}_A \times D_A \), and replaces it with the rather different relation of coherence modulo \( X \), contained in \( |X| \times |X| \). In particular, we don't have \( a \preceq_A y \) iff for all \( x \in A \), \( x \preceq y \pmod{x} \). The relation \( \preceq_A \) is transitive in the sense that

if \( a \preceq_A y \) for all \( y \in b \) and \( b \preceq_A z \) then \( a \preceq_A z \).

We can have \( a \cup b \in X \) and \( b \cup z \in X \), but \( a \cup z \notin X \).
The following auxiliary notions are required.

**Definition 5.0.5:**
Let $X$ be a coherent space, and let $x, x' \in |X|$.

(i) **Strict coherence:**

\[ x \sim x' \, [\text{mod } X] \text{ iff } x \supset x' \, [\text{mod } X] \text{ and } x \neq x'; \]

\[ \text{iff } \{x, x'\} \notin X \text{ and } x \neq x'. \]

(ii) **Strict incoherence:**

\[ x \sim x' \, [\text{mod } X] \text{ iff } \neg \left( x \supset x' \, [\text{mod } X] \right); \]

\[ \text{iff } \{x, x'\} \notin X. \]

(iii) **Incoherence:**

\[ x \sqsubseteq x' \, [\text{mod } X] \text{ iff } \neg \left( x \sim x' \, [\text{mod } X] \right); \]

\[ \text{iff } \{x, x'\} \notin X \text{ or } x = x'. \]

**Definition 5.0.6:**
If $X$ is a coherent space then its **linear negation** $X^\perp$ is defined by

\[ |X^\perp| \overset{df}{=} |X|; \]

\[ x \sqsubseteq x' \, [\text{mod } X^\perp] \text{ iff } x \sqsubseteq x' \, [\text{mod } X]. \]

So linear negation exchanges coherence and incoherence. Note that we have the identity $X^{\perp\perp} = X$. Hence we have a tertium non datur:

for all $x, x' \in |X|$, either $x \supset x' \, [\text{mod } X]$ or $x \sqsubseteq x' \, [\text{mod } X^\perp]$. 

Actually, that is cheating: the above translates as
for all $x,x' \in \Sigma^*$, either $\{x,x'\} \in \Sigma$ or $\{x,x'\} \notin \Sigma$ or $x = x'$.

How about quantum nondatur?

Observe that the objects $\alpha \in \Sigma^*$, which are (the underlying set of) complete subgraphs in $W(\Sigma^*)$, are discrete subgraphs of $W(\Sigma)$, i.e. collections of isolated nodes.
Graphically, the $(-)^*$ operation puts in an edge everywhere there wasn't one and removes all the edges that were there, except the reflexive ones. Recalling our previous examples, the linear negation of a flat coherent space is a round coherent space, and vice-versa. For example, $\text{Int}^*$ is a countable boolean algebra.

**Definition 5.0.7:**

\[
1 \equiv \text{Sgl} : 1 \equiv 1^* ; \quad 0 \equiv \text{Emp} : \quad T \equiv 0^* .
\]

You've already seen the problem: Sgl and Emp are both flat and round.

\[
|1| = \bot = \frac{1}{2} @ \frac{1}{2} , \quad @ \equiv [\text{mod } 1] \quad \text{and} \quad @ \equiv [\text{mod } \bot] ; \\
|0| = \top = \emptyset .
\]

This is not merely a glitch that can be patched up; we just have to live with it. Our dual constants are still going to behave the way we want them to, and we use the distinct notations to indicate this, but underneath the notation they are, in this semantics, the same.
The $\frac{1}{1}$ and $\frac{0}{1}$ difficulty brings to the fore an important point: canonical isomorphisms $\approx$ are not going to correspond to provable equivalence in $\text{LL}$ or $\text{LL}!$ (while identities in the phase semantics do). If two formulae are provably equivalent then the coherent spaces interpreting them will be canonically isomorphic. A counter-example to the converse is that $\top \approx \bot$ (there is nothing more canonical than identity) but, of course, $\top =_{\text{LL}} \bot$ is nonsense. As some consolation, all the sensible canonical isomorphisms will correspond to provable equivalences.

5.1 Direct sums and stable functions

The two additive operations, $\&$ and $\oplus$, are variations on the theme of the direct sum (disjoint union): get it?

Definition 5.1.0:
Let $X$ and $Y$ be coherent spaces.
The coherent space $X \& Y$ is defined as follows:

$|X \& Y| \triangleq |X| + |Y| = 0 \times |X| \cup 1 \times |Y| ;$

$(0,x) \subset (0, x') [\text{mod } X \& Y]$ iff $x \subset X'$ [mod $X$];
$(1,y) \subset (1, y') [\text{mod } X \& Y]$ iff $y \subset Y'$ [mod $Y$];
$(0,x) \subset (1, y) [\text{mod } X \& Y]$ for all $x \in |X|, y \in |Y|.$

The coherent space $X \oplus Y$ is defined as follows:

$|X \oplus Y| \triangleq |X| + |Y| ;$
\[(0, x) \lor (0, x) \equiv_{\text{mod} \times \oplus Y} \text{ iff } x \equiv x' \mod \times; \]
\[(1, y) \land (1, y') \equiv_{\text{mod} \times \oplus Y} \text{ iff } y \equiv y' \mod \times; \]
\[(0, x) \land (1, y) \equiv_{\text{mod} \times \oplus Y} \text{ for all } x \in |X| \text{ and } y \in |Y|.\]

The operation \& puts in all edges \((0, x) \lor (1, y)\) across the divide between \(\{0\} \times |X|\) and \(\{1\} \times |Y|\), while the \(\oplus\) operation leaves \(\{0\} \times |X|\) and \(\{1\} \times |Y|\) totally disconnected. This accords (roughly) with the intuitive story about \& as meaning that both conjuncts are available, and we get to choose which one is realized, while with \(\oplus\), the choice is not ours. Choice is involved in both operations because any given node \((i, z)\) in \(|X \land Y| = |X \lor Y|\) must be in exactly one of \(\{0\} \times |X|\) and \(\{1\} \times |Y|\).

**Observation 5.1.1:** (Girard, Lafont & Taylor [1987])

For all objects \(c \in X \land Y\), there are unique \(a \in X\) and \(b \in Y\) such that \(c = \{0\} \times a \cup \{1\} \times b\); write \(c = a + b\) for short.

For all objects \(c \in X \lor Y\), either \(c = \{0\} \times a\) for some \(a \in X\) or \(c = \{1\} \times b\) for some \(b \in Y\).

**Observation 5.1.2:** (Girard [1987])

Let \(X, Y\) and \(Z\) be coherent spaces.
We have the de Morgan identities

\[(X \land Y)^* = X^* \lor Y^*, \quad (X \lor Y)^* = X^* \land Y^*,\]

and canonical isomorphisms expressing the commutativity and associativity of \& and \(\oplus\), and indicating the status of \(\top\) and \(\bot\) as identities for \& and \(\oplus\) respectively:
\[ X \& Y \equiv Y \& X, \quad X \oplus Y \equiv Y \oplus X, \]
\[ (X \& Y) \& Z \equiv X \& (Y \& Z), \quad (X \oplus Y) \oplus Z \equiv X \oplus (Y \oplus Z), \]
\[ X \& T = X, \quad X \oplus 0 = X. \]

In each case, the canonical isomorphism in question is the most obvious one.

We shall return to \& and \oplus a little later. Our task for the present is an examination of classes of functions on coherent spaces.

**Definition 5.1.3**

Let \( X \) and \( Y \) be coherent spaces.

A function \( F : X \to Y \) is **continuous** iff \( F \) satisfies the following conditions:

(I) monotonicity: if \( a, a' \in X \) and \( a \leq a' \) then \( F(a) \leq F(a') \);

(II') preservation of directed unions (continuity):

if \( S \subseteq X \) is directed then \( F(U^\uparrow S) = U^\uparrow \{ F(a) \mid a \in S \} \).

Note that monotonicity implies that the family \( \{ F(a) \mid a \in S \} \)
in \( Y \) is directed if \( S \subseteq X \) is directed: if \( a \cup b \subseteq c \) then \( F(a) \leq F(c) \) and \( F(b) \leq F(c) \) hence \( F(a) \cup F(b) \leq F(c) \).

Recalling our checklist of properties in Observation 5.0.4, the continuity property (II') above is of course weaker than the preservation of all the unions there are, which was property (II) on that list. The term 'continuity' derives from the theory of domains: when domains are given a topological interpretation, the property of preserving directed unions corresponds to continuity in the
topological sense; see Scott [1982]. Scott also uses the term 'approximable' to refer to a class of functions similar to those we call continuous. That term is most appropriate here: recalling that every object $a \in X$ is the directed union of its finite subsets, continuity implies

$$(\text{FIN}) \quad F(a) = \bigcup \{ F(a_0) \mid a_0 \preceq a \text{ and } a_0 \in X_{\text{fin}} \}$$

So continuous functions are completely determined by their values on the finite objects of $X$. With little work, it can be shown that (FIN) is actually equivalent to the continuity condition ($\mathbb{I}'$).

**Lemma 5.1.4:** (Girard, Lafont and Taylor [1989])

Let $F : X \rightarrow Y$ be continuous.

If $a \in X$ and $y \in F(a)$

then there is a **minimal** finite $a_0 \preceq a$ such that $y \in F(a_0)$, i.e. $a_0$ such that if $b \preceq a_0$ and $y \in F(b)$ then $a_0 = b$.

**Proof:**

From the equation (FIN), we know that if $y \in F(a)$ then $y \in F(b)$ for some finite $b \preceq a$. Now consider all the subsets $c \subseteq b$ such that $y \in F(c)$, and pick one of minimal cardinality. ■

The catch is that, when $y \in F(a)$, there may be more than one minimal finite $a_0 \preceq a$ such that $y \in F(a_0)$. Our aim is to construct a coherent space whose objects are in one-one correspondance with functions $F : X \rightarrow Y$ of some type.
Definition 5.1.5:
Let \( F : X \to Y \) be continuous.
The **trace** of \( F \), denoted \( \text{Tr}(F) \), is defined as follows:

\[
\text{Tr}(F) \overset{\Delta}{=} \{(a,y) \in X_{\text{fin}} \times Y \mid y \in F(a) \text{ and if } b \leq a \text{ and } y \in F(b) \text{ then } a = b \}.
\]

Observe that \( \text{Tr}(F) \) is a set \( f \subseteq X_{\text{fin}} \times Y \) with the property that, for all \( (a,y), (a',y') \in f \),

1. if \( a = a' \) then \( y \equiv y' \mod Y \), and
2. if \( a' \leq a \) and \( y = y' \) then \( a = a' \).

Condition (1) just says that if \( y \in F(a) \) and \( y' \in F(a) \) then \( \forall y, y' \in Y \); this follows because \( \forall y, y' \in F(a) \). Condition (2) is an immediate consequence of the definition of \( \text{Tr}(F) \).

Now it can be shown that, given any set \( f \subseteq X_{\text{fin}} \times Y \) satisfying conditions (1) and (2), we can define a continuous function \( F : X \to Y \) by the formula

\[
\text{(APP)} \quad F(a) = \{ y \in Y \mid \exists a_0 \in F((a_0, y) \in f) \}.
\]

The proof only makes use of condition (1) and the fact that the \( a_0 \) in \( (a_0, y) \in f \) are finite.
Specifically, if \( a_0 \in X_{\text{fin}} \) and \( \{a_i \}_{i \in I} \) is a directed subfamily of \( X \), then if \( a_0 \subseteq \bigcup_{i \in I} a_i \) then \( a_0 \leq a_k \) for some \( k \in I \). Technically, the objects \( a_0 \in X_{\text{fin}} \) are the compact elements of a coherent space \( X \).
Lemma 5.1.6:
Let $F: X \rightarrow Y$ be continuous.
For each $a \in X$, define

$$G(a) = \frac{1}{2} \{ y \in Y \mid \exists a_{0} \in a \ ((a_{0}, y) \in Tr(F)) \}.$$ 

Then $G(a) = F(a)$ for all $a \in X$.

Proof:
Let $y \in G(a)$. Then there is an $a_{0} \subseteq a$ such that

$$(a_{0}, y) \in Tr(F).$$

Hence $y \in F(a_{0})$. Since $F$ is continuous,

$$F(a) = \bigcup \{ F(a_{0}) \mid a_{0} \subseteq a \text{ and } a_{0} \in X_{f_{n}} \}.$$ 

Hence $y \in F(a)$. Conversely, suppose $y \in F(a)$. Then there is (at least one) minimal $a_{0} \subseteq a$ such that $y \in F(a_{0})$. Hence $(a_{0}, y) \in Tr(F)$

and so $y \in G(a)$.

Now we have a one-one correspondence between continuous functions $F: X \rightarrow Y$ and subsets $f \subseteq X_{f_{n}} \times Y$, satisfying conditions (1) and (2), hence $X_{f_{n}} \times Y$ is a good candidate for the domain of the web of the coherent space we're after. However, we still have to ensure that the subsets $f$ are coherent with respect to some coherence relation.

Let $F$ be continuous and suppose both $(a, y) \in Tr(F)$ and $(a', y) \in Tr(F)$. So $y \in F(a)$, $y \in F(a')$ and both $a$ and $a'$ are minimal w.r.t. this property. But that is all we know about $a$ and $a'$ — they may be disjoint; their union may or may not be an element of $X$; we don't know. Basically, we have not got enough information about the sets $a \in X_{f_{n}}$ to define a coherence relation.

To overcome this deficiency, we consider a smaller class of functions. The extra condition we impose, expressing a property called stability, first came to light in Berry [1978] in work on a semantic characterization of sequential algorithms.
Definition 5.1.7:
Let $X$ and $Y$ be coherent spaces.
A continuous function $F : X \to Y$ is called **stable** iff $F$ satisfies the following condition:

**($\text{III'}$)** stability: if $a, a' \in X$ and $a \cup a' \in X$
then $F(a \cap a') = F(a) \cap F(a')$.

The stability condition is a special case of preservation of intersections. By the obvious induction, if $\{a_i : i \in I\}$ is any finite linked family in $X$, and hence $\bigcup_{i \in I} a_i \in X$, then

$$F(\bigcap_{i \in I} a_i) = \bigcap_{i \in I} F(a_i).$$

In category-theoretic terms, the stability condition says that $F$ preserves the pullback

$$
\begin{array}{ccc}
   a \cup a' & \to & a' \\
   \downarrow & & \downarrow \\
   a & \to & a' \\
   \bigcap a & \to & \bigcap a'
\end{array}
$$

where the arrows are inclusions.

Lemma 5.1.8. (Girard, Lafont & Taylor [1989])
Let $F : X \to Y$ be continuous.
Then $F$ is stable iff for each $a \in X$, if $y \in F(a)$ then there is a unique minimal $a_o \in X_{\text{fin}}$ such that $y \in F(a_o)$; equivalently, $a_o = \bigcap \{ b \in X \mid b \subseteq a \text{ and } y \in F(b) \}.$
Proof:
Assume $F : X \to Y$ is stable and suppose $y \in F(a)$. By Lemma 5.1.4, there is a minimal finite $a_0 \in a$ such that $y \in F(a_0)$. Now let $b \leq a$ be such that $y \in F(b)$. Since $a_0 \cup b \leq a$, we have $a_0 \cup b \in X$. Since $F$ is stable, $F(a_0) \cap F(b) = F(a_0 \cap b)$. Hence $y \in F(a_0 \cap b)$.
Now since $a_0$ is minimal, we must have $a_0 = a_0 \cap b$. Hence $a_0 = \bigcap \{ b \in X \mid b \leq a \text{ and } y \in F(b) \}$.

Conversely, suppose $F : X \to Y$ is continuous and for each $a \in X$ with $y \in F(a)$, there is a smallest $a_0 \in a$ such that $y \in F(a_0)$. Let $a, a' \in X$. Then by monotonicity, $F(a \cap a') \subseteq F(a) \cap F(a')$. Suppose now that $a \cup a' \in X$, and let $y \in F(a) \cap F(a')$. By monotonicity, $y \in F(a \cup a')$.

Let $a_0 \leq a \cup a'$ be such that $y \in F(a_0)$ and $a_0 = \bigcap \{ b \in X \mid b \leq a \cup a' \text{ and } y \in F(b) \}$.

Since $y \in F(a)$ and $y \in F(a')$, we have $a_0 \subseteq a$ and $a_0 \subseteq a'$, hence $a_0 \subseteq a \cap a'$. Hence by monotonicity, $y \in F(a \cap a')$.

Exercise 5.1.9:
Prove the following:
If $F : X \to Y$ is stable then

(III') preservation of intersections of linked subfamilies of $X$: if $S \subseteq X$ is linked and non-empty, hence $U \in S \in X$, then $F(\cap S) = \bigcap \{ F(a) \mid a \in S \}$.

Hint: the argument above readily generalizes.
Recall our checklist of properties in Observation 5.0.4. Stable functions are (I) monotone, (II') preserve directed unions, and (III'') preserve intersections of linked subfamilies. What about properties (IV) and (V) on that list? What do stable functions do with singletons and the empty set? The short answer is almost anything they like. For example, a constant function \( F: X \to Y \) defined by \( F(a) = b \) for all \( a \in X \), where \( b \neq \emptyset \), is stable. For each \( a \in X \) and \( y \in F(a) = b \), we have \( y \notin F(\emptyset) = b \), hence \( \emptyset \) is the smallest \( a \leq a \) such that \( y \in F(a) \).

**Lemma 5.1.10** (Girard, Lafont & Taylor [1989])

Let \( F : X \to Y \) be stable.

Then the trace of \( F \),

\[
\text{Tr}(F) = \{ (a, y) \in X \times Y \mid \forall y \in F(a) \text{ and if } b \exists a \text{ and } y \in F(b) \text{ then } a = b \}
\]

has the property that, for all \((a, y), (a', y') \in \text{Tr}(F)\),

1. if \( a = a' \) then \( y \subseteq y' [\text{mod } Y] \), and
2. if \( a = a' \) and \( y = y' \) then \( a = a' \).

**Proof:**

1. We have \( y \in F(a) \) and \( y' \in F(a') \). If \( a = a' \), then \( \forall y, y' \in F(a) \), hence \( \forall y, y' \in Y \).

2. If \( a = a' \) and \( y = y' \), then \( y \in F(a) \). Let \( a_0 \) be the smallest subset of \( a \) such that \( y \in F(a_0) \).

Then \( a_0 \leq a \) and \( a \leq a_0 \). But then by the minimality clause in the definition of \( \text{Tr}(F) \), \( a = a_0 = a' \).
Lemma 5.1.11 (Girard, Lafont and Taylor [1989])

Let \( X \) and \( Y \) be coherent spaces.

If \( f \in X \le X \times |Y| \) is such that, for all \((a,y),(a',y') \in f\),

1. if \( a \cup a' \in X \) then \( y \equiv y' \pmod{Y} \), and
2. if \( a \cup a' \in X \) and \( y = y' \) then \( a = a' \),

then there is a unique stable function \( F : X \to Y \) defined by

\[
(\text{APP}) \quad F(a) = \{ y \in |Y| \mid \exists a_o \in a \ (a_o, y) \in f \}.
\]

Proof:

First, we verify that \( F \) is a function from \( X \) into \( Y \); to show \( F(a) \in Y \), we must check that \( F(a) \) is a coherent subset of \( |Y| \). Let \( y_o, y \in F(a) \). Then there are \( a_o \subseteq a \) and \( a_c \subseteq a \) such that \((a_o, y_o) \in f\) and \((a_c, y) \in f\). But then \( a_o \cup a_c \subseteq a \), so by (1) above, \( y_o \equiv y_c \pmod{Y} \).

Now we have to show that \( F \) is stable.

(i) Monotonicity: if \( a \subseteq a' \) then clearly \( F(a) \subseteq F(a') \).

(ii) Continuity. Let \( \{a_i\}_i \subseteq X \) be a directed subfamily of \( X \).

Put \( a = \bigcup_\iota a_i \), so \( a \in X \). Since \( a_i \subseteq a \) for each \( i \in \iota \), we have \( \bigcup_\iota F(a_i) \subseteq F(a) \) by monotonicity. Conversely, if \( y \in F(a) \) then there is a finite \( a_o \subseteq a \) such that \((a_o, y) \in f\), hence \( y \in F(a_o) \). Now \( a_o \subseteq \bigcup_\iota a_i \), hence \( a_o \subseteq a_k \) for some \( k \in \iota \) since the family \( \{a_i\}_i \subseteq X \) is directed. By monotonicity, \( F(a_o) \subseteq F(a_k) \), hence \( y \in \bigcup_\iota F(a_i) \).
(iii) stability. Suppose \( a \in X \). Then \( F(\{ a \} \cap F(a')) \in F(a) \cap F(a') \) by monotonicity. Let \( y \in F(a) \cap F(a') \). Then there are finite \( a_0 \in a \) and \( a_1 \in a' \) such that \((a_0, y) \in F(a) \cap F(a')\). Now \( a_0 \cup a_1 \in a \cup a' \) hence \( a_0 \cup a_1 \in X \). So by condition (2), \( a_0 = a_1 \). We have \( y \in F(a_0) \) and \( a_0 \in a \cup a' \), hence by monotonicity, \( y \in F(\{ a \} \cup F(a')) \).

Since stable functions are continuous, uniqueness comes from Lemma 5.1.6.

**Definition 5.1.12:**

Let \( X \) and \( Y \) be coherent spaces.

The coherent space \( X \Rightarrow Y \) is defined as follows:

\[ |X \Rightarrow Y| = \frac{|X \cap Y|}{|X \cap Y|}; \]

\((a, y) \subseteq (a', y') \) [mod \( X \Rightarrow Y \)]

iff (1) if \( a \cup a' \in X \) then \( y \subseteq y' \) [mod \( Y \)]; and

(2) if \( a \cup a' \in X \) and \( y = y' \) then \( a = a' \).

**Theorem 5.1.13:** (Girard, Lafont, Taylor [1989])

Let \( X \) and \( Y \) be coherent spaces.

Then

\[ X \Rightarrow Y = \frac{1}{\frac{1}{2}} \text{Tr}(F) \mid F : X \Rightarrow Y \text{ is stable}; \]

**Proof:**

By Lemmas 5.1.10 and 5.1.11.
The intention is that encoding a stable function by its trace corresponds to λ-abstraction and the formula (APP) defines the application of a function. Verification that the coherent semantics developed thus far is sufficient to interpret the positive λ, Π, Σ fragment of intuitionistic propositional calculus, and hence gives a model of typed λ-calculus, comes from the following result.

**Theorem 5.1.14:** (Lafont [1988a])

The category \( \mathbf{St} \) of coherent spaces, with stable functions as its morphisms, is cartesian closed.

**Outline of proof:**

We will not give all the gory details.

Clearly, \( T \) is the terminal object in \( \mathbf{St} \). For each \( X \), there is only one map \( F : X \to T \), given by \( F(a) = \emptyset \) for all \( a \in X \), and this \( F \) is stable.

To ensure that \( \& \) is a product, we need projections. Define \( P_0 : X \& Y \to X \) by \( P_0(\{a_1 \& a_2\}) = a_1 \), and \( P_1 : X \& Y \to Y \) by \( P_1(\{a_1 \& a_2\}) = a_2 \).

(Recall that each object \( c \in X \& Y \) admits a unique decomposition.) By mere inspection, \( P_0 \) and \( P_1 \) are stable functions.

Now let \( g_0 : Z \to X \) and \( g_1 : Z \to Y \) be stable functions. We claim there is a unique stable function \( F : Z \to X \& Y \) such that the following diagram commutes.
We define a set $f \subseteq Z_{fin} \times \lvert X \& Y \rvert$ by

$$f = \{ (c, (0, x)) \mid (c, x) \in Tr(G_0) \} \cup \{ (c, (1, y)) \mid (c, y) \in Tr(G_1) \}$$

We write $(i, w) \in \lvert X \& Y \rvert$, i.e. $i \in \{0, 1\}$ and $w \in \lvert X \rvert \cup \lvert Y \rvert$. Let $(c, (i, w)), (c', (i', w')) \in f$.

1. Suppose $c = c' \in Z$. If $i = i'$ then $(c, w), (c', w') \in Tr(G_i)$ and hence $(i, w) \subseteq (i, w') \pmod{X \& Y}$. If $i \neq i'$ then we have $(i, w) \subseteq (i', w') \pmod{X \& Y}$ automatically.

2. Suppose $c = c' \in Z$ and $(i, w) = (i', w')$. Then $(c, w), (c', w') \in Tr(G_i)$ and hence $c = c'$.

Hence $f$ is the trace of the stable function $F : Z \rightarrow X \& Y$ such that the diagram in question commutes.

By Theorem 5.1.13, the map

$$\text{Tr} : \text{Hom}_{st}(X, Y) \rightarrow (X \Rightarrow Y)$$

is a bijection. By putting a suitable ordering on $\text{Hom}_{st}(X, Y)$ (see Lafont [1988a], Exercise 6) it can be shown that $\text{Tr}$ is an isomorphism between $\text{Hom}_{st}(X, Y)$ and $X \Rightarrow Y$ considered as partial orders. So $\Rightarrow$ is the internal hom on $\text{st}$. To establish closure, we have to exhibit a bijection

$$\text{Hom}_{st}(X \& Y, Z) \equiv \text{Hom}_{st}(X, Y \Rightarrow Z)$$

natural in $X$ and $Z$. A canonical isomorphism

$$(X \& Y) \Rightarrow Z \cong X \Rightarrow (Y \Rightarrow Z)$$

will suffice.
\[(x \& y) \Rightarrow z \mid = (x \& y)_{\text{fin}} \times |z|\]

Note that \(c \in (x \& y)_{\text{fin}}\) iff \(c = \emptyset \times x \cup \emptyset \times y\) for some (uniquely determined) \(a \in x_{\text{fin}}\) and \(b \in y_{\text{fin}}\).

\[|x \Rightarrow (y \Rightarrow z)| = x_{\text{fin}} \times |y \Rightarrow z| = x_{\text{fin}} \times (y_{\text{fin}} \times |z|)\]

It is readily verified that the map

\[(c, z) \mapsto (a, (b, z))\]

is a graph isomorphism and hence induces the required canonical isomorphism of coherent spaces. \(\square\)

It can be shown that the category \(\text{cont}\) of coherent spaces with continuous functions as its morphisms is not Cartesian closed. One supposes that there is an 'exponential' coherent space \(Y^X\) such that

\[\text{Hom}_{\text{cont}}(x, y) \cong Y^x\]

and derives a contradiction. A sketch of the proof is given in Lafont [1988a], (Exercise 2).

Note that the additive conjunction \(\oplus\) is not a co-product on \(\text{St}\). The injections \(J_0 : x \to x \oplus y\) and \(J_1 : y \to x \oplus y\) defined by \(J_0(a) = \emptyset \times a\) and \(J_1(b) = \emptyset \times b\) are stable, and every object \(c \in x \oplus y\) can be written as \(J_0(a)\) for some \(a \in x\) or \(J_1(b)\) for some \(b \in y\). The expression is unique, except in the case of \(\emptyset\); we have \(\emptyset = J_0(\emptyset) = J_1(\emptyset)\). This makes it impossible to define a function \(F : x \oplus y \to z\) case-wise by

\[F(J_0(a)) = a_0(a), \quad F(J_1(b)) = a_1(b)\]
from two stable functions $G_0: X \to Z$ and $G_1: Y \to Z$. The problem is that $G_0(\emptyset)$ has no good reason to be equal to $G_1(\emptyset)$; see Girard, Lafont and Taylor [1989], Chapter 12.

5.2 Linearization: decomposing intuitionistic implication

We have the coherent space $X \Rightarrow Y$, where

$|X \Rightarrow Y| \overset{df}{=} X_{fin} \times |Y|$, and

$(a,y) \subseteq (a',y') \iff \mod X \Rightarrow Y$

iff

(1) if $aua' \in X$ then $y \subseteq y'$ \iff \mod Y; and

(2) if $aua' \in X$ and $y = y'$ then $a = a'$.

Such a coherent space begs to be decomposed into simpler parts. Girard called coherent space semantics 'disturbing'; the disturbance comes from the realization that intuitionistic implication is not primitive.

**Definition 5.2.0:**

Let $X$ be a coherent space.

The coherent space $!X$ is defined as follows:

$|!X| \overset{df}{=} X_{fin}$ and $a \subseteq a' \iff \mod !X$ iff $aua' \in X$.

**Definition 5.2.1:**

Let $X$ and $Y$ be coherent spaces.

The coherent space $X \Rightarrow Y$ is defined as follows:

$|X \Rightarrow Y| \overset{df}{=} |X| \times |Y|$. 
\[(x, y) \equiv (x', y') \pmod{X \rightarrow Y}\]

iff (1) if \(x \equiv x' \pmod{X}\) then \(y \equiv y' \pmod{Y}\);
(2) if \(x \equiv x' \pmod{X}\) and \(y = y'\) then \(x = x'\).

In the presence of (1), (2) is equivalent to
(2') if \(x \equiv x' \pmod{X}\) then \(y \equiv y' \pmod{Y}\).

Evidently, \(X \Rightarrow Y = \neg X \rightarrow Y\).

**Exercise 5.2.2:**
Fill in the blank:
Stable functions are to \(\Rightarrow\) as \(\_\_\_\_\_\) functions are to \(\rightarrow\).

**Definition 5.2.3:**
Let \(X\) and \(Y\) be coherent spaces.
A stable function \(F: X \rightarrow Y\) is called \underline{linear} iff \(F\)
satisfies the following condition:

(II) \underline{linearity:} if \(S \subseteq X\) is linked
then \(F(US) = \bigcup \{ F(a) \mid a \in S \} \).

So linear functions preserve all the unions there are. By
taking \(S = \emptyset\) in (II), we get (IV'): \(F(\emptyset) = \emptyset\). Note that
there is nothing to rule out \(F(a) = \emptyset\) where \(a \neq \emptyset\) (eg. the
maps \(F: X \rightarrow T\) so (IV') is weaker than (IV) in 5.0.4). The
linearity condition (II) implies continuity (IV') and monotonicity
(I), so in view of our earlier discussion (Exercise 5.1.9),
we have the following equivalence:

\(F: X \rightarrow Y\) is linear
iff for all linked subfamilies \(S \subseteq X\),
(II) \(F(US) = \bigcup \{ F(a) \mid a \in S \}\), and
(III') \(F(\cap S) = \bigcap \{ F(a) \mid a \in S \}\), provided \(S\) is non-empty.
Lemma 5.2.4: (Girard, Lafont and Taylor [1989])
Let \( F: X \to Y \) be Stable.

\( F \) is linear

\[ \text{iff} \quad \text{Tr}(F) = \{ (tx,y) \in X_{\text{fin}} \times Y \mid y \in F(tx) \} \]

**Proof:**

Suppose \( F: X \to Y \) is linear and for some \( y \in Y \),
\( (a, y) \in \text{Tr}(F) \) but \( a \) is not a singleton. Suppose \( a = \emptyset \).

then \( y \in F(\emptyset) \neq \emptyset \), contradicting linearity. So assume \( a \neq \emptyset \).

then for any \( a' \) such that \( \emptyset \subseteq a' \subseteq a \) with \( a' \neq \emptyset \) and
\( a' \neq a \), we must have \( y \notin F(a') \), since \( a' \) is minimal.

Now \( ava' = a \), so \( ava' \in X \). Hence we have
\( F(ava') = F(a) \neq F(a) \cup F(a') \), contradicting linearity.

Conversely, suppose \( F: X \to Y \) is stable and
\( \text{Tr}(F) = \{ (tx,y) \in X_{\text{fin}} \times Y \mid y \in F(tx) \} \). Let \( S \subseteq X \)
be any linked subfamily. By monotonicity, we have
\( U \{ F(a) \mid a \in S \} \subseteq F(US) \). Let \( y \in F(US) \). Then there
is a unique \( x \in US \) such that \( y \in F(tx) \). Now \( x \in a \)
for some \( a \in S \), hence \( y \in F(a) \subseteq U \{ F(a) \mid a \in S \} \).

Now if we know that a function \( F \) is linear, we can safely discard the singleton symbols in \( \text{Tr}(F) \).

**Definition 5.2.5:**

Let \( F: X \to Y \) be linear.

The **linear trace** of \( F \), denoted \( \text{Tr}_{\text{lin}}(F) \), is defined as follows:

\[ \text{Tr}_{\text{lin}}(F) \triangleq \{ (x,y) \in X \times |X| \times Y \mid y \in F(tx) \} \]
**Lemma 5.2.6:** (Girard, Lafont & Taylor [1989])

Let \( F : X \to Y \) be linear.

Then \( \text{Tr} \text{lin}(F) \) has the property that, for all \( (x,y), (x',y') \in \text{Tr} \text{lin}(F) \),

1. if \( x \equiv x' \pmod{X} \) then \( y \equiv y' \pmod{Y} \), and
2. if \( x \equiv x' \pmod{X} \) and \( y = y' \) then \( x = x' \).

**Proof:**

Recalling that \( \{ x \uplus u \uplus x' \} = \{ x, x' \} \in X \) iff \( x \equiv x' \pmod{X} \),
the result follows immediately from Lemma 5.1.10.

**Lemma 5.2.7:** (Girard, Lafont & Taylor [1989])

Let \( X \) and \( Y \) be coherent spaces.

If \( f \subseteq \{ X \times X \} \) is such that for all \( (x,y), (x',y') \in f \),

1. if \( x \equiv x' \pmod{X} \) then \( y \equiv y' \pmod{Y} \), and
2. if \( x \equiv x' \pmod{X} \) and \( y = y' \) then \( x = x' \),

then there is a unique linear function \( F : X \to Y \)

defined by

\[
\text{(LIN-APP)} \quad F(a) = \{ y \in Y \mid \exists x \in a \quad ((x,y) \in f) \}
\]

**Proof:**

Analogous to the proof of Lemma 5.1.11. The only extra item to check is the linearity condition. Let \( S \subseteq X \)
be linked. We have \( U \{ F(a) \mid a \in S \} \subseteq F(US) \) by the monotonicity of \( F \). Let \( y \in F(US) \). Then there is an \( x \in US \)
such that \( (x,y) \in f \). Hence \( y \in F(\{x\}) \). Now \( x \in a \) for some \( a \in S \), hence \( y \in F(a) \subseteq U \{ F(a) \mid a \in S \} \).
Theorem 5.2.8: (Girard, Lafont and Taylor [1989])
Let $X$ and $Y$ be coherent spaces.
Then
$$X \rightarrow Y = \{ \text{Trin}(F) \mid F: X \rightarrow Y \text{ is linear} \}.$$ 

Proof: by Lemmas 5.2.6 and 5.2.7.

When $F: X \rightarrow Y$ is linear then, given $a \in X$ and $y \in F(a)$, there is a unique $x \in a$ such that $y \in F(x)$. Note, however, that linear functions need not preserve singletons: we can have $y \in F(x)$ and $y \in F(x')$, provided $y \equiv y' \mod Y$ (condition (1)). So linear functions do not in general give rise to graph homomorphisms on the webs. Condition (2) is like a local injectivity property: if $y \in F(x)$ and $y \in F(x')$ and $x \neq x'$, then $x \sim x' \mod X$.

![Diagram](image)

Lemma 5.2.9: (Girard [1987])
Let $X$ and $Y$ be coherent spaces.
We have the following canonical isomorphisms:
$$X \rightarrow Y \approx Y^+ \rightarrow X^+,$$
$$1 \rightarrow X \approx X, \quad X \rightarrow \perp \approx X^+.$$

Proof:
(i) The map \((x,y) \mapsto (y,x)\) gives a graph isomorphism \(W(X \to Y) \cong W(Y \to X^*)\).
(ii) The maps \((@, x) \mapsto x\) and \((x, @) \mapsto x\) give graph isomorphisms \(W(\bot \to X') \cong W(X)\) and \(W(X \to \bot) \cong W(X^*)\) respectively; explicitly:
\[ (@, x) C (\bar{x}, x) \mod \bot \to X \]
iff:
1. \(\bar{x} \circ @ \mod \bot\) then \(x \circ x' \mod X\), and
2. \(x \circ x' \mod X\) then \(@ \circ \bar{x} \mod \bot\).

Recall the other component of intuitionistic implication decomposed: coherence spaces \(!X\) where \(\|X\| \equiv X_{\text{fin}}\) and \(a \circ a' \mod \|X\|\) iff \(a a' \in X\). The objects of \(!X\) are linked subfamilies \(S \subseteq X_{\text{fin}}\), i.e. for all \(a, a' \in S\), \(a a' \in X\). For each object \(a \in X\), define
\[ !a \equiv \{ a_0 \in X_{\text{fin}} \mid a_0 \subseteq a \}. \]
So \(!a \in !X\). This construction defines a function \(F : X \to !X\) given by \(F(a) = !a\). Note that \(F\) is stable, but not linear; a quick computation reveals that \(\text{Tr}(F) = \{ (a_0, a_0) \mid a_0 \in X_{\text{fin}} \}\).

We have a general procedure for turning stable functions into linear ones, and vice-versa.

Lemma 5.2.10: (Girard, Lafont & Taylor [1989])
If \(F : X \to Y\) is stable
then there is a unique linear function \(\text{Lin}(F) : !X \to Y\)
defined by the equation \(\text{Trin}(\text{Lin}(F)) = \text{Tr}(F)\).
Proof:
We show that \( \text{Lin}(F)(!a) = F(a) \).

Let \( y \in F(a) \). Then there is a unique finite \( a_0 \leq a \) such that \( (a_0, y) \in \text{Tr}(F) = \text{TrLin}(\text{Lin}(F)) \). Hence \( y \in \text{Lin}(F)(!a_0) \). Since \( !a_0 \leq !a \), we have \( y \in \text{Lin}(F)(!a) \) by monotonicity. The argument that \( y \in \text{Lin}(F)(!a) \) implies \( y \in F(a) \) is similar.

**Corollary 5.2.11:** (Girard, Lafont & Taylor [1989])

If \( G : !X \to Y \) is linear then there is a unique stable function \( \text{Delin}(G) : X \to Y \) defined by \( \text{Delin}(G)(a) = G(!a) \) and satisfying the equation \( \text{Tr}(\text{Delin}(G)) = \text{TrLin}(G) \).

Proof:
The operations \( \text{Lin} \) and \( \text{Delin} \) are mutual inverses.

Let \( \text{Lin} \) denote the category of coherent spaces with linear functions as its morphisms. Technically, the previous two results say that the functor \( ! : \text{Lin} \to \text{Lin} \) is the left adjoint to the forgetful functor from \( \text{Lin} \) into \( \text{St} \). Moreover, \( ! \) is a comonad (or cotriple): there are families of linear functions \( E_X : !X \to X \) and \( D_X : !X \to !!!X \) that do the sorts of things they are supposed to do. The functor \( ! \) has other pleasant properties which we'll say a little more about later.

Of course, there is the dual of \( ! : \) the functor? is defined so as to ensure that we have identities
\[
(!X)^* = ?X^* \quad \text{and} \quad (?X)^* = !X^*.
\]
Definition 5.2.11:
Let \( X \) be a coherent space.
The coherent spaces \( !X \) and \( ?X \) are defined as follows:

\[
|!X| \triangleq X_{\text{fin}} \quad \text{and} \quad a \triangleq a'[\text{mod } !X] \iff au a' e X \\
|?X| \triangleq (X^\dagger)_{\text{fin}} \quad \text{and} \quad a \triangleq a'[\text{mod } ?X] \iff au a' e X^\dagger
\]

(The definition of \( !X \) is repeated for comparison purposes.)

Observe that \( a \triangleq a'[\text{mod } ?X^\dagger] \iff a \triangleq a'[\text{mod } ?X] \)
\( \iff au a' e X^\dagger \iff a \triangleq a'[\text{mod } X^\dagger] \). Similar computations confirm the other de Morgan identity.

Fact 5.2.12: (Lafont [1988a])
The category \( \text{Lin} \) is Cartesian and co-cartesian. The product is \( \& \), as in \( \text{St} \); the projections \( P_0 \) and \( P_1 \) are linear. The co-product is \( \oplus \); the problem which prevents \( \oplus \) being a co-product in \( \text{St} \) is now fixed because \( G(\emptyset) = \emptyset \) for all linear functions \( G \). Emp = \( T = \emptyset \) is both terminal and initial.

Using the co-product \( \oplus \) in \( \text{Lin} \), we can define a co-product structure on the category \( \text{St} \) by performing a "linearization". For coherent spaces \( X \) and \( Y \), define

\[
X \oplus Y \triangleq !X \oplus !Y
\]

So \( |X \oplus Y| = \{0\} X_{\text{fin}} \cup \{1\} Y_{\text{fin}} \) and

\[
(0, a) \triangleq (0, a') \quad \text{[mod } X \oplus Y] \quad \text{iff} \quad au a' e X, \\
(1, b) \triangleq (1, b') \quad \text{[mod } X \oplus Y] \quad \text{iff} \quad bu b' e Y, \quad \text{and} \\
(0, a) \triangleq (1, b) \quad \text{[mod } X \oplus Y] \quad \text{for all } a e X_{\text{fin}} \text{ and } b e Y_{\text{fin}}.
\]
The injections $K_0 : X \to X \uplus Y$ and $K_1 : Y \to X \uplus Y$ given by $K_0(a) = \{0\} \times a$ and $K_1(b) = \{1\} \times b$ are stable. Given stable functions $G_0 : X \to Z$ and $G_1 : Y \to Z$, we can define $F : X \uplus Y \to Z$ casewise:

- For $S \in \{X, Y\}$, $F(\{0\} \times S) = \text{Lin}(G_0)(S)$, and
- For $T \in \{X, Y\}$, $F(\{1\} \times T) = \text{Lin}(G_1)(T)$.

In particular, $F(\emptyset) = \text{Lin}(G_0)(\emptyset) = \text{Lin}(G_1)(\emptyset) = \emptyset$.

$F$ is linear, and hence stable. And finally, the initial object in $\mathbf{St}$ is $\emptyset = \text{Emp}$, as in $\mathbf{Lin}$.

**Fact 5.2.13:** (Lafont [1988a])

$$\text{TrLin} : \text{Hom}_{\mathbf{Lin}}(X, Y) \cong (X \leadsto Y)$$

By Theorem 5.2.8, we know that $\text{TrLin}$ is a bijection; the verification that it is an isomorphism of partial orders is similar to that required for $\Rightarrow$.

The category $\mathbf{Lin}$ is not, however, Cartesian closed.

We don't have

$$(X \& Y) \to Z \cong X \leadsto (Y \to Z)$$

What closure there is, is with respect to the monoidal structure on $\mathbf{Lin}$.

**Definition 5.2.14:**

Let $X$ and $Y$ be coherent spaces.

The coherent spaces $X \otimes Y$ and $X \& Y$ are as follows:

- $|X \otimes Y| \equiv |X| \times |Y|$;
- $(x, y) \equiv (x', y') [\mod X \otimes Y]$ iff $x \equiv x' [\mod X]$ and $y \equiv y' [\mod Y]$;
- $(x, y) \equiv (x', y') [\mod X \& Y]$ iff $x \equiv x' [\mod X]$ or $y \equiv y' [\mod Y]$.
Fact 5.2.15: (Lafont [1988a])

The operation $\otimes$ defines on $\text{Lin}$ the structure of a symmetric monoidal category, with $1$ as the unit.

What this means is that $\otimes$ is a bifunctor and there are canonical (natural) isomorphisms

$$X \otimes 1 \cong X \quad X \otimes Y \cong Y \otimes X \quad (X \otimes Y) \otimes Z \cong X \otimes (Y \otimes Z).$$

Satisfying the Mac Lane–Kelly equations (see Maclane [1971]).

Fact 5.2.16: (Lafont [1988a])

The category $\text{Lin}$ is a closed symmetric monoidal category; i.e. there is a natural bijection

$$\text{Hom}_{\text{Lin}}(X \otimes Y, Z) \cong \text{Hom}_{\text{Lin}}(X, Y \to Z),$$

which means

$$(X \otimes Y) \to Z \cong X \to (Y \to Z).$$

Fact 5.2.17: (Seely [1989])

The category $\text{Lin}$ is a $\ast$-autonomous category (in the sense of Barr [1979]), which means it is a closed symmetric monoidal category together with an involution $(-)\ast: \text{Lin}^{\text{op}} \to \text{Lin}$, isomorphisms $X \to Y \to Y^\ast \to X^\ast$ and isomorphisms $X \to X^{\ast\ast}$.

Following Seely [1989], we have

Definition 5.2.18:

A linear category is a $\ast$-autonomous category with finite products.
Note that if a \(*\)-autonomous category has a product then it has a coproduct by de Morgan duality.

Not unexpectedly, \textbf{Lin} is a linear category. The category \textbf{Vect}_k of finite dimensional vector spaces over a field \(k\), with linear maps as its morphisms, is also a linear category. The duality operator \((-)^\ast\) is the involution, the monoidal structure is given by the tensor product \(\otimes\) and the coproduct is the direct sum \(\oplus\). Unfortunately, \textbf{Vect}_k is not a terribly good categorical model of propositional linear logic: in \textbf{Vect}_k we have

\[(V \oplus W)^\ast \cong V^\ast \oplus W^\ast, \quad (V \otimes W)^\ast \cong V^\ast \otimes W^\ast.\]

So in \textbf{Vect}_k, \(\oplus\) and \(\&\) are collapsed into one, and likewise for \(\otimes\) and \(\&\). Recent work by Lafont, cited in Martí-Oliet and Meseguer [1989], seeks to generalize the category \textbf{Vect}_k. Lafont defines a category \textbf{Games}_k, for any set \(k\), and shows that \textbf{Lin} is isomorphic to a full subcategory of \textbf{Games}_{10,11}.

\begin{fact}
(Seely [1989])
The category \textbf{Lin} together with the comonad \(!: \text{Lin} \to \text{Lin}\) is such that

(i) for each \(X\) of \textbf{Lin}, \(!X\) is a comonoid with respect to the monoidal structure;

(ii) there are natural isomorphisms \(!X \otimes !Y \cong !(X \& Y)\) and \(1 \cong !T\), and all the appropriate diagrams commute.
\end{fact}
Following Seely [1989], any linear category with a comonad satisfying (i) and (ii) above (with the details suitably filled in) is called a Girard category.

Seely takes the (categorical) essence of Girard's translation of intuitionistic logic into linear logic to be the following.

**Proposition 5.2.20:** (Seely [1989])

If \( G, ! \) is a Girard category

then the Kleisli category \( K(G) \) with respect to \( ! \)

is Cartesian closed.

In particular, \( St \) is the Kleisli category of \( Lin, ! \).

Seely notes that in general, \( K(G) \) does not have co-products. \( St \) is a special case because \( Lin \)

is a subcategory of \( St \).

A much more detailed discussion of the categorical interpretation of linear logic is to be found in Seely [1989]. See also Marti-Oliet and Meseguer [1989] and Lafont [1988a].

The general procedure in categorical logic is to associate, under the Lambek-Lawvere correspondence, morphisms of the category in question with equivalence classes of proofs in a given logical system, the equivalence classes being given by the defining equations of the category. In the case of linear categories, a rather long list of equations is required; see Seely [1989] and Marti-Oliet and Meseguer [1989];
A presentation in terms of categorical combinators is given in Lafont [1988b].

Girard takes a different course. In the next sub-section we see a direct construction of a semantic object for each proof in the sequent calculus $\mathbb{LL}_r$. The equivalence classes of proofs determined by this construction are of importance in the normalization (cut-elimination) process for the system of proof nets dealt with in Section 6.

5.3 Proofs as objects in coherent spaces.

Fix a countable collection of coherent spaces (excluding $\text{Sgl}$ and $\text{Emp}$) and a bijective mapping of that collection onto the set of propositional letters of $\mathcal{L}_i$, the language of modal linear logic. Assign the coherent space $\text{Sgl}$ to both the constants $\top$ and $\bot$, and the coherent space $\text{Emp}$ to $T$ and $\emptyset$. The duals of propositional letters and all compound formulae built from $\emptyset$, $\&$, $\oplus$, $\sharp$, $!$, and $?$ will have assigned to them the appropriate coherent space formed from the semantic operations $(\cdot)^\perp$, $\otimes$, $\&$, $\oplus$, $\#$, $!$, and $?$. Henceforth, we identify each formula of $\mathcal{L}_i$ with its coherent space.

If $\Delta$ is a sequence of formulae $A_1, \ldots, A_n$ then $\Delta = A_1 \otimes \ldots \otimes A_n$ as a coherent space. Hence

$$|\Delta| = |A_1| \times \ldots \times |A_n|.$$

We write $Z \in |\Delta|$ where $Z = (z_1, ..., z_n)$ and $z_i \in |A_i|$ for $i = 1, ..., n$. From the definition of the operation $\&$ on coherent spaces (Definition 5.2.14), we have

$Z \sim Z' \mod \Delta$ if $z_i \sim z_i' \mod A_i$ for some $i \in \{1, ..., n\}$,

and $Z \subseteq Z' \mod \Delta$ if $Z = Z'$ or $Z \sim Z' \mod \Delta$.

We shall define by induction a (unique) object $\Pi^* \in \Delta$ for each proof $\Pi$ in $\mathbb{L}$ of $\vdash \Delta$. That $\Pi^*$ represents the proof $\Pi$ should become clear from the construction. As an example, we'll be able to show that

if $\Pi_0$ is a proof of $\vdash A$,

$\Pi_i$ is a proof of $\vdash A \Rightarrow B$ (i.e. $\vdash A^* \& B$),

and $\Pi_i^* = \text{Trin}(F)$ where $F : A \Rightarrow B$ is linear

then $F(\Pi_0^*) = \Pi^*$ where $\Pi$ is the proof obtained from $\Pi_0$ and $\Pi_i$ by applying (Cut).

Theorem 5.3.0: (Girard [1987])

If $\Pi$ is a proof of $\vdash \Delta$ in $\mathbb{L}$,

then there is a unique object $\Pi^* \in \Delta$ which represents $\Pi$.

Proof:

The object $\Pi^*$ is defined by induction on the proof $\Pi$ of $\vdash \Delta$. In each case, it will be obvious that $\Pi^* \subseteq |\Delta|$. What has to be verified is that for all $Z \in \Pi^*$ and $Z' \in \Pi^*$, $Z \sim Z' \mod \Delta$. We use commas to denote the concatenation of sequences and omit...
Outermost parentheses where appropriate.

(i) \( \Pi \) is an instance of the identity axiom scheme, say
\( \vdash A, A^\dagger \).
Define \( \Pi^* \equiv \{ (x, x) \mid x \in A \} \).
Recall that for all \( x \in A \) and \( x' \in A \) with \( x \neq x' \),
we have either \( x \not\leftrightarrow x' \) [mod A] or \( x \leftrightarrow x' \) [mod A^*].
Hence for all \( (x, x) \) and \( (x', x') \in \Pi^* \), \( (x, x) \equiv (x', x') \) [mod A^\dagger A^\dagger].

(ii) \( \Pi \) is obtained from \( \Pi_0 \) by an application of (EXCH):
\[
\frac{\vdash \Gamma}{\vdash \sigma(\Gamma)} \quad \text{(EXCH)}
\]
Where \( \sigma \) is a permutation of \( \Gamma \).
Define \( \Pi^* \equiv \{ \sigma(\tau) \mid \tau \in \Pi_0^* \} \).
Assuming \( \Pi_0^* \) is coherent, so is \( \Pi^* \).

The additives:

(iii) \( \Pi \) is an instance of the axiom scheme for \( T \), say
\( \vdash T, \Gamma \).
Define \( \Pi^* \equiv \emptyset \), since \( |T| \times |\Gamma| = \emptyset \times |\Gamma| = \emptyset \).

(iv) \( \Pi \) is obtained from \( \Pi_0 \) and \( \Pi_i \) by the rule (\&):
\[
\frac{\vdash A, \Gamma \quad \vdash B, \Gamma}{\vdash A \& B, \Gamma} \quad \text{(\&)}
\]
Define \( \Pi^* \equiv \{(0, x), Z_0 \mid x, Z_0 \in \Pi_0^* \} \cup \{(1, y), Z_1 \mid y, Z_1 \in \Pi_i^* \} \).
Now \((0, x), Z_0 \equiv (0, x'), Z_0' \) [mod A\&B, \Gamma] since by hypothesis,
we have \( x, Z_0 \equiv x', Z_0' \) [mod A, \Gamma]; likewise for pairs of elements
of \( \Pi^* \) obtained from \( \Pi_i^* \). Since \((0, x) \equiv (1, y) \) [mod A\&B]
for all \( x \in A \) and \( y \in B \), we have \((0, x), Z_0 \equiv (1, y), Z_1 \) [mod A\&B, \Gamma].
(v) \( \Pi \) is obtained from \( \Pi_0 \) by the rule \((\text{fst } \Theta)\):
\[
\frac{}{\vdash A, \Gamma} \quad (\text{fst } \Theta)
\]
\[
\vdash A \otimes B, \Gamma
\]
Define \( \Pi^* \equiv \{ (0, x), z \mid x, z \in \Pi_0^* \} \).

The coherence of \( \Pi^* \) follows immediately from the coherence of \( \Pi_0^* \).

(vi) \( \Pi \) is obtained from \( \Pi_i \) by the rule \((\text{snd } \Theta)\):
\[
\frac{}{\vdash B, \Gamma} \quad (\text{snd } \Theta)
\]
\[
\vdash A \otimes B, \Gamma
\]
Define \( \Pi^* \equiv \{ (1, y), z \mid y, z \in \Pi_i^* \} \).

(vii) \( \Pi \) is the axiom \( \vdash \bot \).
Define \( \Pi^* \equiv \{ @ \} \).

(viii) \( \Pi \) is obtained from \( \Pi_0 \) by the rule \((\bot)\):
\[
\frac{}{\vdash \Gamma}
\]
\[
\vdash \bot, \Gamma
\]
Define \( \Pi^* \equiv \{ @, z \mid z \in \Pi_0^* \} \).

(ix) \( \Pi \) is obtained from \( \Pi_0 \) and \( \Pi_i \) by the rule \((\otimes)\):
\[
\frac{}{\vdash A, \Pi_0} \frac{}{\vdash B, \Pi_i} \quad (\otimes)
\]
\[
\vdash A \otimes B, \Pi_0, \Pi_i
\]
Define \( \Pi^* \equiv \{ (x, y), z_0, z_i \mid x, z_0 \in \Pi_0^* \text{ and } y, z_i \in \Pi_i^* \} \).

Let \( \omega = (x, y), z_0, z_i \) and \( \omega' = (x', y'), z_0', z_i' \).

By hypothesis, we have \( x, z_0 \prec x', z_0' [\text{mod } A, \Gamma] \) and \( y, z_i \prec y', z_i' [\text{mod } B, \Gamma] \).

Let \( \Delta \) be an abbreviation for \( A \otimes B, \Pi_0, \Pi_i \).
We argue by cases.

Case (1): \( x, z_0 = x', z_0' \) and \( y, z_1 = y', z_1' \). Then \( W = W' \).

Case (2): \( x, z_0 \sim x', z_0' \) [mod \( A, \Gamma_0 \)] and \( y, z_1 = y', z_1' \).

Now either \( z_0 \sim z_0' \) [mod \( \Gamma_0' \)], in which case \( W \sim W' \) [mod \( \Delta \)],
or else \( x \sim x' \) [mod \( A \)]. Since \( y = y' \), we have
\( (x, y) \sim (x', y') \) [mod \( A \otimes B \)], hence \( W \sim W' \) [mod \( \Delta \)].

Case (3): \( x, z_0 = x', z_0' \) and \( y, z_1 \sim y', z_1' \) [mod \( B, \Gamma_1 \)].
The argument is symmetric with case (2).

Case (4): \( x, z_0 \sim x', z_0' \) [mod \( A, \Gamma_0 \)] and \( y, z_1 \sim y', z_1' \) [mod \( B, \Gamma_1 \)].

Then either \( x \sim x' \) [mod \( A \)] and \( y \sim y' \) [mod \( B \)],
or \( z_0 \sim z_0' \) [mod \( \Gamma_0 \)] or \( z_1 \sim z_1' \) [mod \( \Gamma_1 \)];
in all cases, we have \( W \sim W' \) [mod \( \Delta \)].

(x) \( \Pi \) is obtained from \( \Pi_0 \) by the rule (8):
\[
\frac{\vdash A, B, \Gamma}{\vdash A \otimes B, \Gamma} \quad (8)
\]
Define \( \Pi^* \overset{d}{=} \Pi_0^* \).

The exponentials:

(xi) \( \Pi \) is obtained from \( \Pi_0 \) by the rule (Tn?):
\[
\frac{\vdash \Gamma}{\vdash ?A, \Gamma} \quad (Tn?)
\]
Define \( \Pi^* \overset{d}{=} \{ \phi, \xi \mid \xi \in \Pi_0^* \} \).

Coherence is immediate: \( \phi, \xi \sim \phi, \xi' \) [mod \( ?A, \Gamma \)] iff
\( \xi \sim \xi' \) [mod \( \Gamma \)].

(xii) \( \Pi \) is obtained from \( \Pi_0 \) by the rule (C?):
\[
\frac{\vdash ?A, ?A, \Gamma}{\vdash ?A, \Gamma} \quad (C?)
\]
Define \( \Pi^* \overset{d}{=} \{ a \cup b, \xi \mid a, b, \xi \in \Pi_0^* \text{ and } a \cup b \in A^* \} \).
Let \( a \cup b, z \) and \( a' \cup b', z' \in \Pi^* \). By hypothesis, we have
\[ a, b, z \in a', b', z' \mod \Pi \] and \( a \cup b \in A^+ \) and \( a' \cup b' \in A^+ \).
Hence either (1) \( a, b, z = a', b', z' \mod \Pi \) or (2) \( z \approx z' \) [mod \( \Pi \)]
or (3) \( a \approx a' \mod \Pi \) or (4) \( b \approx b' \mod \Pi \). In cases (1) and (2), we have \( a \cup b, z \in a' \cup b', z' \mod \Pi \).
Assume either (3) or (4) obtains. Then either \( a \cup a' \notin A^+ \) or \( b \cup b' \notin A^+ \). Hence \( (a \cup b) \cup (a' \cup b') \notin A^+ \), for otherwise we would have \( a \cup a' \in A^+ \) and \( b \cup b' \in A^+ \) by down closure. Hence \( (a \cup b) \cup (a' \cup b') \mod \Pi \) and so \( a \cup b, z \in a' \cup b', z' \mod \Pi \).

(Xiii) \( \Pi^* \) is obtained from \( \Pi_0 \) by the rule (D?):
\[
\frac{\vdash A, \Pi}{\vdash ? A, \Pi} \quad (D?)
\]
Define \( \Pi^* = \{ \{x, z \mid x, z \in \Pi_0 \} \}
Let \( \{x, z \} \) and \( \{x', z' \} \in \Pi^* \). By hypothesis, we have
\[ x, z \in x', z' \mod \Pi \] Hence either (1) \( x, z = x', z' \) or (2) \( x \approx x' \mod \Pi \) or (3) \( x \approx x' \mod A \). In cases (1) and (2), we have \( \{x, z \} \subseteq \{x', z' \} \mod \Pi \). Assume (3) obtains. Then \( x \approx x' \mod A^+ \) hence \( \{x, x' \} \notin A^+ \), and so \( \{x, z \} \approx \{x', z' \} \mod \Pi \).

(Xiv) \( \Pi^* \) is obtained from \( \Pi_0 \) by the rule (!):\
\[
\frac{\vdash A, B_1, \ldots, B_k}{\vdash ! A, B_1, \ldots, B_k} \quad (!)
\]
Define \( \Pi^* \) to be the set of all sequences of the form
\[ \{x_1, \ldots, x_n\}, z \cup \ldots \cup z \in \Pi^* \] such that
(a) \( n > 1 \);
(b) \( \{x_1, \ldots, x_n\} \in A \);
(c) for \( i = 1, \ldots, n \), \( x_i, z_i \in \Pi^* \), where each \( z_i \) is of the form \( (b_{i1}, \ldots, b_{ik}) \) and \( b_{ij} \in (B_j^+) \) for \( j = 1, \ldots, k \).
(d) for $j = 1, \ldots, k$, $b_3 u \ldots u b_n \in B^*_1$. We write $z_1u \ldots u z_n \in \Gamma^*$ for short.

We may assume the $x_1, \ldots, x_n$ are distinct: it can be shown that $x_1 = x_2$ implies $z_1 = z_2$, so the repetition does not add anything and a size $(n-1)$ sequence would do just as well.

Now let $\tilde{\xi} = \{x_1, \ldots, x_n\}, z_1u \ldots u z_n$ and $\tilde{\eta} = \{y_1, \ldots, y_m\}, \omega_1u \ldots u \omega_m$ be elements of $\Pi^*$. We argue by cases.

Case (1): $\tilde{\xi} = \tilde{\eta}$. Then $\tilde{\xi} \cup \tilde{\eta} [\mod A, ?\Gamma]$.

Case (2): $\tilde{\xi} \neq \tilde{\eta}$ and $\{x_1, \ldots, x_n\} \cup \{y_1, \ldots, y_m\} \in A$. Then $\{x_1, \ldots, x_n\} \cup \{y_1, \ldots, y_m\} [\mod A]$ hence $\tilde{\xi} \cup \tilde{\eta} [\mod A, ?\Gamma]$.

Case (3): $\tilde{\xi} \neq \tilde{\eta}$ and $\{x_1, \ldots, x_n\} \cup \{y_1, \ldots, y_m\} \notin A$. Then there must be an $x_i$ and a $y_j$ such that $\{x_i, y_j\} \notin A$, hence $x_i \sim y_j [\mod A]$. But since $x_i, z_i \in \Pi_0^*$ and $y_j, \omega_j \in \Pi_0^*$, we have $x_i, z_i \cup y_j, \omega_j [\mod A, ?\Gamma]$.

Hence $z_i \sim \omega_j [\mod ?\Gamma]$. This in fact implies that $z_1u \ldots u z_n \sim \omega_1u \ldots u \omega_m [\mod ?\Gamma]$. Hence $\tilde{\xi} \cup \tilde{\eta} [\mod A, ?\Gamma]$.

We have not forgotten the (Cut) rule. We have left it until last because the definition of $\Pi^*$ in this case involves an unbounded existential quantifier; in all the other cases, the definition of $\Pi^*$ uses only elementary set-theoretic operations applied to $\Pi_0^*$ and $\Pi_i^*$, where $\Pi_0, \Pi_i$ are subproofs.

(XV) $\Pi$ is obtained from $\Pi_0$ and $\Pi_i$ by the rule (Cut):

\[
\frac{\vdash A, \Gamma_0 \quad \vdash A, \Gamma_i}{\vdash \Gamma_0, \Gamma_i} \quad \text{(Cut)}
\]

Define $\Pi^*\equiv \{z_0, z_i \mid \exists x \in \text{Al} (x, z_0 \in \Pi_0^* \text{ and } x, z_i \in \Pi_i^*)\}$. 

Suppose \( z_0, z \in \Pi^* \) with witness \( x \in \Pi \) and \( y_0, y \in \Pi^* \) with witness \( y \in \Pi \). By hypothesis, we have \( x, z_0 \equiv y, y_0 \ [\mod A, \Gamma] \) and \( x, z \equiv y, y \ [\mod A^+, \Gamma^+] \). Now either (i) \( x \equiv y \ [\mod A] \) or (2) \( x \equiv y \ [\mod A^+] \).

If case (1) holds, we must have \( z_0 \equiv y_0 \ [\mod \Gamma] \), and if case (2) holds, we have \( z \equiv y \ [\mod \Gamma^+] \). Either way, we get \( z_0, z \equiv y, y \ [\mod \Gamma, \Gamma^+] \).

A consolation is that the existential quantifier has uniqueness built in; cf. the existential quantifier in the formula \((\text{APP})\) by which a linear (or stable) function is recovered from its trace.

Suppose \( x, z_0 \in \Pi_0^* \) and \( y, y_0 \in \Pi_0^* \) and \( x, z \in \Pi_1^* \) and \( y, y \in \Pi_1^* \). Then \( x, z_0 \equiv y, z_0 \ [\mod A, \Gamma^+] \) and \( x, z \equiv y, z \ [\mod A^+, \Gamma^+] \). The first implies that either \( x \equiv y \) or \( x \equiv y \ [\mod A] \); the second implies that either \( x \equiv y \) or \( x \equiv y \ [\mod A^+] \). Since we can't have both \( x \equiv y \ [\mod A] \) and \( x \equiv y \ [\mod A^+] \), we must have \( x \equiv y \).

This completes the construction.

Now we verify our earlier claim. Suppose \( \Pi_0 \) is a proof of \( \vdash \Pi \), \( \Pi_1 \) is a proof of \( \vdash A \rightarrow B \) and \( \Pi_1^* = \text{Tr} \Pi_0 \Pi_1 \) where \( F: A \rightarrow B \) is linear. Now both as formulae and as coherent spaces, \( A \rightarrow B = A^+ \& B \).

We may assume the proof \( \Pi_1 \) of \( \vdash A^+ \& B \) is obtained from a proof \( \Pi_2 \) of \( \vdash A^+, B \) by the rule \((8)\). Then \( \Pi_2^* = \Pi_1^* = \text{Tr} \Pi_0 \Pi_1 \). Hence
\[ F(T_0^*) = \{ y \in B \mid \exists x \in T_0^* ((x,y) \in T \text{ lin}(F)) \} \]

\[ = \{ y \in B \mid \exists x \in A \{ x \in T_0^* \text{ and } (x,y) \in T_2^* \} \} \]

Let \( T \) be the proof of \( \Gamma \vdash \beta \) obtained from the proof \( T_0 \) of \( \Gamma \vdash \alpha \) and the proof \( T_2 \) of \( \Gamma \vdash \alpha^* \), \( \beta \) by (Cut). Then \( T^* = F(T_0^*) \).

This is all in accordance with Heyting's ideal: the proof \( T_1 \) of \( \Gamma \vdash \alpha \rightarrow \beta \) is, in an appropriate sense, a function which takes a proof \( T_0 \) of \( \Gamma \vdash \alpha \) to a proof \( F(T_0) \), i.e. \( T_1 \), of \( \Gamma \vdash \beta \).

**Observation 5.3.1:**

We say that two proofs \( T_0 \) and \( T_i \) of a sequent \( \Gamma \vdash \Delta \) differ only by their order of rules iff (i) \( T_0 \) and \( T_i \) have the same occurrences of instances of axioms schemes; and (ii) for all rules (R) of LL! other than (EXCH), (R) is applied in \( T_0 \) with principal formula \( A \) and premises formulae \( B \) and \( C \) (or premises formula \( B \)) iff (R) is applied in \( T_i \) with the same principal and premises formulae.

If \( T_0 \) and \( T_i \) are two proofs of \( \Gamma \vdash \Delta \) which differ only by their order of rules, then \( T_0^* = T_i^* \).

This observation gives us some information about the equivalence relation

\[ T_0 \sim T_i \iff T_0^* = T_i^* \]

defined for proofs \( T_0 \) and \( T_i \) in LL! of a given sequent \( \Gamma \vdash \Delta \).
When else do two proofs \( \Pi_0 \) and \( \Pi_1 \) of \( \vdash \Delta \) correspond to the same semantic object?

Consider the following two configurations of proofs.

(1)

\[
\begin{align*}
\Pi_0 \quad & \vdash A^\perp, A \quad \text{(8)} \\
[\mu] \vdash A^\perp \otimes A \quad & \text{(8)} \\
[\mu] \vdash A^\perp \otimes A^\perp, \Delta_0, \Delta_1 \quad & \text{(cut)} \\
[\mu] \vdash \Delta_0, \Delta_1
\end{align*}
\]

(2)

\[
\begin{align*}
\Pi_0 \quad & \vdash A, \Delta_0 \quad \text{cut} \\
\Pi_1 \quad & \vdash A^\perp, \Delta_1 \quad \text{cut} \\
[\Pi] \vdash \Delta_0, \Delta_1
\end{align*}
\]

The proof \( \Pi \) of \( \vdash \Delta_0, \Delta_1 \) is obtained from the proof \( \mu \) of \( \vdash \Delta_0, \Delta_1 \) by 'pushing the cut upwards'; this is one of a number of possible conversions in the procedure for eliminating cuts. Now compute the semantics:

\[
\Pi^* = \{ (x, z, z) \mid \exists x \in \text{AL} \ (x, z, z \in \Pi_0^* \text{ and } x, z, z \in \Pi_1^*) \}
\]

\[
\mu_0^* = \{ (u, u) \mid u \in \text{AL} \}.
\]

\[
\mu_1^* = \{ (u, v), w_0, w_1 \mid u, w_0 \in \Pi_0^* \text{ and } v, w_1 \in \Pi_1^* \}.
\]

\[
\mu^* = \{ (u, v), w_0, w_1 \mid \exists (u, v) \in \text{AL} \times \text{AL} \ ((u, v) \in \mu_0^* \text{ and } (u, v), w_0, w_1 \in \mu_1^*) \}
\]

\[
= \{ (u, v), w_0, w_1 \mid \exists u \in \text{AL} \ ((u, v) \in \mu_0^* \text{ and } (u, v), w_0, w_1 \in \mu_1^*) \}
\]

\[
= \{ (u, v), w_0, w_1 \mid \exists u \in \text{AL} \ ((u, w_0 \in \Pi_0^* \text{ and } u, w_1 \in \Pi_1^*) \}
\]

\[
= \Pi^*
\]

This is not a coincidence.
The system of proof nets for linear logic, which is set out in the next section, is something of a hybrid of sequent calculi and natural deduction systems. Its rules are structurally similar to those of LL!, and by an analogous construction, we can define a semantic object $\beta^*$ for each proof net $\beta$ in the system of proof nets PNI!. A key feature of the normalization (cut-elimination) procedure for PNI! is that if $\beta$ is obtained from $\beta_0$ by a sequence of one or more conversions, then $\beta_0^* = \beta^*$. Moreover the map $N$, which sends each proof $\Pi$ in LL! of $\Gamma \vdash A_1, \ldots, A_n$ to a (unique) proof net $N(\Pi)$ with conclusions $A_1, \ldots, A_n$, is surjective. In addition, if $N(\Pi) = \beta$ then $\Pi^* = \beta^*$.

So we can start with a proof $\Pi_0$ of $\Gamma \Delta$ in LL!, then take $\beta_0 = N(\Pi_0)$ and apply any sequence of conversions to $\beta_0$ to get a normal (cut-free) proof net $\beta_0^*$ (guaranteed by the Strong Normalization Theorem for PNI!). By surjectivity, there is a proof $\Pi$ of $\Gamma \Delta$ in LL! such that $\beta = N(\Pi)$; since $\beta$ is cut-free, so is $\Pi$. Then we have

$$\Pi_0^* = \beta_0^* = \beta^* = \Pi^*.$$ 

The eager reader, tantalized by this advertisement for Section 6, may wish to proceed forthwith in search of the joys of proof nets. But hold on; there is something more we can say about the objects $\Pi^* \in \Delta$, where $\Pi$ is a proof of $\Gamma \Delta$: they are big. If $X$ is a coherent space, an object $c \in X$ is maximal iff

$$\forall x \in X (\text{if } \forall y \in c (x \equiv y [\text{mod} x]) \text{ then } x \in c);$$

equivalently, for all $a \in X$, if $c \leq a$ then $a = c$. 
Theorem 5.3.2: 
If $\Pi$ is a cut-free proof of $\vdash \Delta$ in LL then $\Pi^* \in \Delta$ is maximal.

Proof:
By induction on the proof $\Pi$ of $\vdash \Delta$

(i) $\Pi$ is an instance of the identity axiom scheme, say $\vdash A, A^t$.
$\Pi^* = \{ (x, x) \mid x \in \Lambda_1 \}$. 
Let $(y, z) \in \Lambda_1 \times \Lambda_1$. and suppose that for all $x \in \Lambda_1$,
$(x, x) \in (y, z) \mod (A, A^t)$. Take $x = y$. Then $y \equiv z \mod (A^t)$. 
Take $x = z$. Then $z \in y \mod A$. Hence $y = z$ and $(y, y) \in \Pi^*$.

(ii) $\Pi$ is obtained from $\Pi_0$ by the rule (EXCH).
$\Pi^* = \{ \sigma(z) \mid z \in \Pi_0 \} \cup \{ (y, z) \mid y \equiv z \mod A \}$. 
The maximality of $\Pi^*$ follows immediately from that of $\Pi_0^*$.

The additives:

(iii) $\Pi$ is an instance of the axiom scheme for $T$, say $\vdash T, \Gamma$.
$\Pi^* = \emptyset$, and $| T \& \Gamma | = \emptyset$.

(iv) $\Pi$ is obtained from a proof $\Pi_0$ of $\vdash A, \Gamma$ and a proof $\Pi_1$ of $\vdash B, \Gamma$ by the rule $(\&)$.
$\Pi^* = \{ (0, x), z_0 \mid x, z_0 \in \Pi_0^* \} \cup \{ (1, y), z_1 \mid y, z_1 \in \Pi_1^* \}$.
Let $(i, v), w \in | A \& B, \Gamma |$ be such that
(0) for all $x, z_0 \in \Pi_0^*$, $(i, v), w \equiv (0, x), z_0 \mod (A \& B, \Gamma)$, and
(1) for all $y, z_1 \in \Pi_1^*$, $(i, v), w \equiv (1, y), z_1 \mod (A \& B, \Gamma)$.

Suppose $i = 0$, hence $v \in | A |$. Then (0) implies that for all $x, z_0 \in \Pi_0^*$, $v, w \equiv z_0 \mod (A, \Gamma)$. By hypothesis, $\Pi_0^*$
is maximal. So $v, w \in \Pi_0^*$ and hence $\langle 0,v \rangle, w \in \Pi^*$. Likewise, if $i=1$ and hence $\nu \in 1B1$, (i) implies that $v, w \in \Pi_1^*$ and so $\langle 1,v \rangle, w \in \Pi^*$.

(v) $\Pi$ is obtained from a proof $\Pi_0$ of $\vdash A, \Gamma$ by the rule $\text{Fst}$. $\Pi^* = \{ \langle 0, x \rangle, z \mid x, z \in \Pi_0^* \}$

Let $\langle i, v \rangle, w \in \vdash A \Theta B, \Gamma$ be such that for all $x, z \in \Pi_0^*$,

$\langle i, v \rangle, w \in \vdash x, z \ [\text{mod } A \Theta B, \Gamma]$. If $i=0$ then we have $v, w \in \vdash x, z \ [\text{mod } A, \Gamma]$ for all $x, z \in \Pi_0^*$. Since $\Pi_0^*$ is maximal, $v, w \in \Pi_0^*$ and hence $\langle 0,v \rangle, w \in \Pi^*$. If $i=1$ then, since $\langle 1,v \rangle \sim \langle 0, x \rangle \ [\text{mod } A \Theta B]$, we must have $\omega \sim z \ [\text{mod } \Gamma]$ for all $z$ for which there is an $x \in A1$ such that $x, z \in \Pi_0^*$. But now by choosing any $y \in A1$ such that for all $x, z \in \Pi_0^*$, $y \neq x$, we can create an element $\langle y, w \rangle \in A, \Gamma$ which is coherent with every $x, z \in \Pi_0^*$ but not in $\Pi_0^*$; contradiction. So the case $i=1$ is not possible.

(vi) $\Pi$ is obtained from a proof $\Pi_1$ of $\vdash B, \Gamma$ by the rule $\text{Snd}$. $\Pi^* = \{ \langle 1, y \rangle, z \mid y, z \in \Pi_1^* \}$

The argument is symmetric with (v).

The multiplicatives:

(vii) $\Pi$ is the axiom $\vdash 1$. $\Pi^* = \{ @y \}$ and $\{1\} = \{ @y \}$.

(viii) $\Pi$ is obtained from a proof $\Pi_0$ of $\vdash \Gamma$ by the rule $(\bot)$. $\Pi^* = \{ @z, z \mid z \in \Pi_0^* \}$.

The maximality of $\Pi^*$ follows immediately from that of $\Pi_0^*$. 
(ix) \( \Pi \) is obtained from a proof \( \Pi_0 \) of \( \Gamma \vdash A, \delta \), and a proof \( \Pi_i \) of \( \Gamma \vdash B, \delta \), by the rule (8).
\[ \Pi^* = \{ (x, y), z_0, z_i \mid x, z_0 \in \Pi_0^* \text{ and } y, z_i \in \Pi_i^* \} \]

Let \((u, v), w_0, w_i \in \vdash A \otimes B, \delta, \Pi \) be such that for all \((x, y), z_0, z_i \in \Pi^*\), we have \((u, v), w_0, w_i \in \Pi^*\).

Where \( A \) is short for \( A \otimes B, \delta, \Pi \). We argue by cases.

Case (1): \((u, v), w_0, w_i \in \Pi_0^* \) and \((u, v), w_0, w_i \in \Pi_1^* \). Then \((u, v), w_0, w_i \in \Pi^* \).

Case (2): \((u, v), w_0, w_i \in \Pi_0^* \). Since \( \Pi_0^* \) is maximal, there is some \( x, z_0 \in \Pi_0^* \) such that \((u, v), w_0, x, z_0 \in \mod A, \Gamma \). Hence \( u \equiv x \mod A \) and \( w_0 \equiv z_0 \mod \Gamma \). Hence for all \( y, z_i \in \Pi_i^* \), we have
\[
\Pi^* = \{ (x, y), z_0, z_i \mid x, z_0 \in \Pi_0^* \text{ and } y, z_i \in \Pi_i^* \}.
\]

Case (2.1): \((u, v), w_0, w_i \in \Pi_0^* \). Then since \( \Pi_0^* \) is maximal, there exists a \( y, z_i \in \Pi_i^* \) such that \((u, v), w_0, y, z_i \in \mod B, \Gamma \). Hence we have a \( y, z_i \in \Pi_i^* \) such that \( w_0 \equiv z_i \mod \Gamma \), contradicting the conclusion of (2).

Case (2.2): \((u, v), w_0, w_i \in \Pi_1^* \). Then by the conclusion of (2), we have \( w_0 \equiv w_i \mod \Gamma \); contradiction.

Taking on faith the advertised result that semantic objects remain constant under the elimination of cuts, we have the immediate

**Corollary 5.3.3:**

If \( \Pi \) is any proof in \( LL \) of \( \Gamma \vdash A \), then \( \Pi^* \in \Delta \) is maximal.
Those with less faith may demand a direct proof that the rule (CUT) preserves maximality. The author has laboured long and hard on this one, without success.

The four modal rules in LL! are another story. It is relatively easy to show that modal thinning (Th?) and modal contraction (C?) preserve maximality of semantic objects but (D?) and (!) are rather more tricky.

Exercise 5.3.4:
Either prove or find a counterexample to the following propositions.

(i) If \( \Pi \) is a proof of \( \vdash \neg A, \Gamma \) obtained from a proof \( \Pi_0 \)
of \( \vdash A, \Gamma \) by the rule (D?) and \( \Pi_0^* \in A \# \Gamma \) is maximal
then \( \Pi^* \in \neg A \# \Gamma \) is maximal.

(ii) If \( \Pi \) is a proof of \( \vdash \neg ! A, \neg B_1, \ldots, \neg B_k \) obtained from
a proof \( \Pi_0 \) of \( \vdash A, \neg B_1, \ldots, \neg B_k \) by the rule (!)
and \( \Pi_0^* \in A \# B_1 \# \ldots \# B_k \) is maximal
then \( \Pi^* \in ! A \# B_1 \# \ldots \# B_k \) is maximal.

Note: (ii) does hold in the case when \( \Pi \) is a proof of
\( \vdash \neg ! A \) obtained from a proof \( \Pi_0 \) of \( \vdash A \) by (!); in that case,
\( \Pi^* = ! \Pi_0^* = \{ a \in A_{\#} \mid a \in \Pi_0^* \} \).
6. Proof nets and normalization

Girard's system of proof nets is motivated by the desire for a system in which the process of reducing proofs to normal form is reasonably neat and simple. Since linear logic has an involutive negation, normalization in a natural deduction system is going to involve messiness similar to that which arises in classical natural deduction systems. (For a Prawitz-style natural deduction system for linear logic, see Avron [1988].) Proof nets are graphical constructions which are like sequent calculus proofs in that they have multiple conclusions and, moreover, each of the sequent calculus rules is directly mimicked in the system of proof nets; in particular, a normal proof net is one which does not contain any instance of the analogue of the cut rule. (For earlier work on graphical proof systems for multiple conclusion logics, see Shoesmith and Smiley [1978].) On the other hand, the system of proof nets PNI! is a quotient of the sequent calculus LL! in the same way that the intuitionistic natural deduction system NJ is a quotient of the sequent calculus LJ: sequent calculus proofs which differ only by the order of rules are identified. So, in a sense, proof nets are the real proof objects.

With regard to nomenclature, the system we call PNI! is called PNI in Girard [1987]; the system we call PNI, which captures all of propositional linear logic, doesn't have a name in Girard [1987]. Girard goes on to consider PN2, the system with propositional quantifiers; we don't
deal with PN2. We will work through (and fill in the details for) most of the material on proof nets in Girard [1987]. Up to, but not including, the proof of the Strong Normalization Theorem. Primed by this guide, interested readers should be able to tackle that one on their own.

6.0 Tramtrips and PNO

Following Girard, we start by developing a system of proof nets PNO which will match the pure multiplicative sequent calculus LLM (consisting of the identity axiom scheme and the rules (EXCH), (CUT), (⊗) and (⊔)).

Definition 6.0.0:
A proof structure is an object consisting of consisting of occurrences of formulae in \( \mathcal{L}_0 \) (the \( \otimes, \otimes' \) fragment) and links between these occurrences of formulae. The links are of the following kinds:

(i) axiom link: \[ \begin{array}{c}
A \quad A^+ \\
\hline
\end{array} \]
   where \( A \) is a literal.

(ii) cut link: \[ \begin{array}{c}
A \quad A^+ \\
\hline
\text{CUT}
\end{array} \]

(iii) \( \otimes \) link: \[ \begin{array}{c}
A \quad B \\
\hline
A \otimes B
\end{array} \]

(iv) \( \otimes' \) link: \[ \begin{array}{c}
A \quad B \\
\hline
A \otimes' B
\end{array} \]
An axiom link has no premises and two conclusions. With respect to the other kinds of links, occurrences of a formula above the line are premises of the link, and an occurrence of a formula below the line is the conclusion of the link. The symbol 'CUT' is not a formula (so CUT links have two premises and no conclusion). Both axiom links and CUT links are considered symmetric, i.e. A and A⁺ can be interchanged.

We also require that

(1) every occurrence of a formula in the structure is the conclusion of exactly one link; and

(2) every occurrence of a formula in the structure is the premiss of at most one link.

An occurrence of a formula in a proof structure which is not the premiss of any link is called terminal.

A link in a proof structure is called a terminal link of that proof structure iff either its conclusion (or conclusions) is (are) terminal, or else it is a CUT link.

To be precise, we should define an occurrence of a formula as an ordered pair \((A,i)\) where \(A\) is a formula and \(i\) is a positive integer. And, technically, we should define binary relations \(\text{AX}((A,i),(A⁺,j))\) and \(\text{CUT}((A,i),(A⁺,j))\) and ternary relations \(\otimes((A,i),(B,j),(A\otimes B,k))\) and \(\otimes((A,i),(B,j),(A\otimes B,k))\), but that would be needless precision. We shall be content with graphical representations and where there is no risk of ambiguity, speak of formulae rather than formula occurrences.
Examples 6.0.1:
The following diagrams represent proof structures.

(a)

```
  A   B
  \hline
  A \& B
  \hline
  A^+ \& B^+
```

(b)

```
  A   B
  \hline
  A \& B
  \hline
  A^+ \& B^+
```

(c)

```
  A   A^+
  \hline
  CUT
```

Definition 6.0.2:
A proof structure with terminals $A_1, \ldots, A_n$ is correct if
the sequent $\vdash A_1, \ldots, A_n$ is provable in LLM.

Clearly, example (a) above represents a correct proof
structure while examples (b) and (c) do not (the empty
sequent is not provable). So we need a criterion by
which we can distinguish the correct proof structures
from the incorrect. Noting the graphical similarity of
examples (a) and (b), the criterion must be such that
and 8 links can be clearly differentiated. Girard calls his criterion the long trip condition. For pedagogical purposes, we elaborate on the theme of travel, specifically, tram travel.

Definition 6.0.3:
Each proof structure determines a network of tram tracks. With each formula occurrence A in a proof structure, we associate two tram stops: the left hand side tram stop, from which one catches north-bound (→) trams, is denoted A⁺, and the right hand side tram stop, from which one catches south-bound (→) trams, is denoted A⁻.

The layout of track in the network is such that every trip is a round trip; when a tram stops at, say, A⁺, then for some positive integer k, the k⁺th stop to which the tram travels after leaving A⁺ will be A⁻ again.

Let S be the collection of all tram stops in the network and let p be the number of formula occurrences in the corresponding proof structure, so the cardinality of S is 2p. A trip in the network is a bijective function t: S₀ → ℤ/kℤ, where S₀ ⊆ S (S₀ ≤ 2p), subject to some further conditions set out below. We say
two trips \( t : S_0 \rightarrow \mathbb{Z}/k\mathbb{Z} \) and \( t' : S_0 \rightarrow \mathbb{Z}/k\mathbb{Z} \) are equivalent iff for all tram stops \( \xi \in S_0 \),
\[
t'(\xi) = t(\xi) + r \pmod{k}
\]
for some \( k \in \{0, 1, \ldots, k-1\} \). Otherwise put, two trips are equivalent when they define the same cyclic ordering (modulo \( k \)) on \( S_0 \), but differ in the choice of initial tram stop. Two trips are distinct iff they are not equivalent.

We define simultaneously the network of tram tracks for a proof structure and the possible trips through that network by reference to the links and terminal formula occurrences contained in the given proof structure. The definition is component-wise.

(1) **axiom links:**
Tram tracks in the vicinity of an axiom link are as follows.

![Diagram of tram tracks](image)

The tracks are such that every trip \( t \) satisfies
(i) \( t(A^v) = t(A^v^+) + 1 \), and
(ii) \( t(A^+v) = t(A^v) + 1 \).
2. **CUT links:**
Tram tracks in the vicinity of a CUT link are as follows.

![Diagram of CUT link]

The tracks are such that every trip $t$ satisfies
(i) $t(A^\uparrow) = t(A^+_v) + 1$, and
(ii) $t(A^{++}) = t(A_v) + 1$.

3. **terminal formula occurrences:**
Tram tracks in the vicinity of terminal formula occurrences
are as follows.

![Diagram of terminal formula]

The track is such that every trip $t$ satisfies
$t(A^\uparrow) = t(A_v) + 1$.

4. **$\otimes$ links:**
Tram tracks in the vicinity of $\otimes$ links are as follows.

![Diagram of $\otimes$ link]
Associated with each $\otimes$ link is a switch with two positions, "L" and "R".

When the switch is on "L", trips $t$ satisfy the following Conditions:

(i) $t(A^\prime) = t(B^\prime) + 1$;
(ii) $t(B^\prime) = t(A \otimes B^\prime) + 1$;
(iii) $t(A \otimes B^\prime) = t(A^\prime) + 1$.

When the switch is on "R", trips $t$ satisfy the following Conditions:

(i) $t(A^\prime) = t(A \otimes B^\prime) + 1$;
(ii) $t(B^\prime) = t(A^\prime) + 1$;
(iii) $t(A \otimes B) = t(B^\prime) + 1$.

(5) 8 links:

Tram tracks in the vicinity of 8 links are as follows.
Associated with each 8 link is a switch with two positions, "L" and "R".

When the switch is on "L", trips $t$ satisfy the following conditions:

(i) $t(A^\uparrow) = t(A \& B^\uparrow) + 1$;
(ii) $t(B^\uparrow) = t(B_v) + 1$;
(iii) $t(A \& B_v) = t(A_v) + 1$.

When the switch is on "R", trips $t$ satisfy the following conditions:

(i) $t(A^\uparrow) = t(A_v) + 1$;
(ii) $t(B^\uparrow) = t(A \& B^\uparrow) + 1$;
(iii) $t(A \& B_v) = t(B_v) + 1$.

In view of the above specifications, each trip $t$ in the given tram network/proof structure is uniquely determined by the choice of tramstop $\Xi$ such that $t(\Xi) = 0$ and the settings of all the $\Theta$ and $\Theta$ switches in the network. Given a tramstop $\Xi \in S$ and a configuration of $\Theta$ and $\Theta$ switches, let $k \leq 2p$ be the smallest positive integer such that $t(\Xi) = k$, and let $S_0 \subseteq S$ be the collection of all tram stops visited in the course of the trip. In this case, $t$ is a bijection $t: S_0 \rightarrow \mathbb{Z}/k\mathbb{Z}$ (and all the identity symbols in the above specifications are to be reinterpreted as congruence modulo $k$).
Let \( t : S_0 \to \mathbb{Z}/k\mathbb{Z} \) be any trip in the given train network / proof structure. \( t \) is a long trip iff \( k = 2p \); equivalently, iff \( S_0 = S \). Otherwise, \( t \) is a short trip.

(End of Definition 6.0.3.)

Observe that if a proof structure contains \( n \) \( \otimes \) or \( \& \) links then, for any given train stop \( S_i \) in the corresponding train network, there are \( 2^n \) distinct trips \( t \) such that \( S_i \in \text{dom}(t) \).

Examples 6.0.4:
Recall the proof structures represented in 6.0.1. We draw them again, now enhanced with train tracks, and take a few trips.

(a)

The shaded track represents the trips with both the \( \otimes \) switch and the \( \& \) switch set on "R"; the twelve equivalent trips (one for each possible initial train stop) are long trips. The reader is invited to verify that the other three configurations of switch settings also give rise to long trips.
The shaded track corresponds to the trip starting from, say, $A^\perp$ with both \textcircled{X} switches set on "R" (and to the other five equivalent trips). So we have a short trip in an incorrect proof structure.

As expected, we also get a short trip in this proof structure.

\textbf{Definition 6.0.5:}
A proof structure is called a \textit{proof net} iff every trip in the Kan network determined by that proof structure is a long trip. Let $\text{PNO}$ denote the collection of all
proof nets built out of axiom links, cut links, \& links and \&\& links.

Of course, we can’t yet claim that a proof structure is correct iff it is a proof net; the rest of 6.0 will be devoted to establishing this result. Since a proof structure with \( n \) switches give rise to \( 2^n \) distinct trips (which include a given ram stop), the long trip criterion is not a feasible method of checking correctness. In practice, we can always work with proof structures that come directly (by the method set out below) from sequent calculus proofs, or that come from correct proof structures by means of transitions which preserve correctness. Indeed, it is easy to inductively define a collection of correct proof structures (which will turn out to be identical to \( PNO \)) every element of which comes from a proof in \( LLM \). To prove, rather than obtain by definitional theft, that \( PNO \) is what we want it to be, we need an abstract property like the long trip criterion.

On the practical side of things, some improvement has been made by Danos and Regnier [1989]. By working with an alternative graphical representation of proof structures, they show how correctness can be verified by checking at most \( 2^m \) cases, where \( m \) is the number of \&\& links in the given structure. In that paper, Danos and Regnier also develop a generalized multiplicative connective.
Theorem 6.0.6: (Girard [1987])

If $\Pi$ is a proof of $\vdash A_1, \ldots, A_n$ in LLM
then we can naturally associate with $\Pi$ a proof net
$N(\Pi)$ in PNO whose terminal formula occurrences
are exactly $A_1, \ldots, A_n$.

Proof:
$N(\Pi)$ is defined by induction on the proof $\Pi$.

(i) $\Pi$ is an instance $\vdash A, A^+$ of the identity axiom scheme.
Obviously, $N(\Pi)$ is the proof structure

$$
\begin{array}{c}
\Box \\
A & A^+
\end{array}
$$

which is a proof net: all trips are of the form
$\ldots, A^, A^+, A^{++}, A^+, \ldots$, which are long trips.

(ii) $\Pi$ is obtained from $\Pi_0$ by the rule (EXCH).
Set $N(\Pi) = N(\Pi_0)$.

(iii) $\Pi$ is obtained from $\Pi_0$ and $\Pi_1$ by the rule (CUT):

$$
\frac{\vdash A, \Delta_0, \vdash A^+, \Delta_1}{\vdash \Delta_0, \Delta_1} \quad (\text{CUT})
$$

By the induction hypothesis, we have proof nets $N(\Pi_0)$ and $N(\Pi_1)$
which have as their terminals $A$ and all the formula occurrences
in $\Delta_0$, and $A^+$ and all of $\Delta_1$, respectively. We construct
a proof structure $N(\Pi)$ as follows:

$$
\begin{array}{c}
N(\Pi_0) \\
N(\Pi_1)
\end{array}
\quad
\begin{array}{c}
\Box \\
A & A^+
\end{array}
\quad
\begin{array}{c}
\text{CUT}
\end{array}
$$
The set of terminals of $N(\Pi)$ is $\Delta_0 \cup \Delta_1$ (considering $\Delta_0$ and $\Delta_1$ as sets of formula occurrences rather than sequences of formulae). Suppose there are $p$ formula occurrences in $N(\Pi_0)$ and $q$ formula occurrences in $N(\Pi_1)$. Arbitrarily select positions for all the switches in $N(\Pi_0)$ and $N(\Pi_1)$. Let $t_0$ be the resulting trip in $N(\Pi_0)$ starting at $A^\circ$ and let $t_1$ be the trip in $N(\Pi_1)$ starting at $A^{+\circ}$, both to and from are long trips. Let $t$ be the trip in $N(\Pi)$, starting at $A^\circ$, determined by the given configuration of switches. Then $t$ is such that
\[
t(\Sigma_0) = t_0(\Sigma_0) \quad \text{for all train stops } \Sigma_0 \text{ in } N(\Pi_0), \text{ and}
\]
\[
t(\Sigma_1) = 2p + t_1(\Sigma_1) \quad \text{for all train stops } \Sigma_1 \text{ in } N(\Pi_1).
\]

In particular, $t(A^{+\circ}) = 2p + 2q - 1$, and after $A^{+\circ}$ the train must return to $A^\circ$. Hence $t$ is a long trip. Since the switch settings were arbitrary, $N(\Pi)$ is a proof net.

(iv) $\Pi$ is obtained from $\Pi_0$ and $\Pi_1$ by the rule (8):

\[
\frac{\vdash A, \Delta_0 \quad \vdash B, \Delta_1}{\vdash A \oplus B, \Delta_0, \Delta_1}(8)
\]

From proof nets $N(\Pi_0)$ and $N(\Pi_1)$ whose sets of terminals are $\{A\cup \Delta_0\}$ and $\{B\cup \Delta_1\}$ respectively, construct a proof structure $N(\Pi)$, with terminals $\{A \oplus B \cup \Delta_0 \cup \Delta_1\}$ as follows:

\[
N(\Pi_0) \quad \vdash A \quad \vdash B \quad N(\Pi_1)
\]

\[A \oplus B\]

Arbitrarily select positions for all the switches in $N(\Pi_0)$ and $N(\Pi_1)$. Let $t_0$ be the resulting (long) trip in $N(\Pi_0)$ starting at $A^\circ$ and let $t_1$ be the resulting (long)
trip in \( N(\Pi_i) \) starting at \( B^\wedge \). For the given configuration of switches in \( N(\Pi_0) \) and \( N(\Pi_i) \), there are two possible trips in \( N(\Pi) \) starting at \( A^\wedge \): one for each of the two positions of the final \( \otimes \) switch.

If the switch is on "L", the train follows the route of \( t_0 \) until it gets to \( A^v \), then goes to \( A \otimes B_r \), \( A \otimes B^v \) and then \( B^\wedge \). Then it follows the route of \( t_1 \) until it gets to \( B_v \) and then goes back to \( A^\wedge \). Graphically, the situation is as follows:

\[
\begin{array}{c}
\text{If the switch is on "R", then the trip is as follows:}
\end{array}
\]

By inspection, both trips are long. Hence \( N(\Pi) \) is a proof net.

(i) \( \Pi \) is obtained from \( \Pi_0 \) by the rule (8):

\[
\begin{align*}
& \vdash A, B, \Delta \\
& \vdash A \otimes B, \Delta \quad (8)
\end{align*}
\]
From the proof net $N(\Pi_0)$ whose set of terminals is \{A, B\} \cup \Delta_0, construct a proof structure $N(\Pi)$, with terminals \{A\&B\} \cup \Delta_0 as follows:

![Diagram of proof structure](attachment:image.png)

Arbitrarily select positions for all the switches in $N(\Pi_0)$ and let to be the resulting (long) trip in $N(\Pi_0)$ starting at $A^\wedge$. For the given configuration of switches in $N(\Pi_0)$, there are trips in $N(\Pi)$ starting at $A^\wedge$.

If the final $\&$ switch is on "L", the team starts at $A^\wedge$ following the route of to, gets to Bv and loops around to $B^\wedge$ as it does on to, and continues on to Av. Then it goes down to A\&Bv, around to A\&B$^\wedge$ and back to $A^\wedge$.

If the final $\&$ switch is on "R", the team follows the route of to until it gets to Bv, then goes down to A\&Bv, around to A\&B$^\wedge$ and up to $B^\wedge$. From $B^\wedge$ onwards to Av, the team resumes the course of to. After Av, it loops back to $A^\wedge$, as it does on to.

Graphically, the two cases are as follows:

![Diagram of two cases](attachment:image.png)
In both cases we have long trips. Hence \( N(T_{i0}) \) is a proof net.

**Proposition 6.0.7 (Girard [1987]):**

If \( T_{i0} \) and \( T_{i1} \) are two proofs of \( \vdash A \) in \( LLM \) which differ only by their order of rules, then \( N(T_{i0}) = N(T_{i1}) \).

The proof of 6.0.7 is just an elaboration of the fact that sequent calculus proofs proceed sequentially, adding one new connective at a time, while proof nets are built up in a parallel fashion. As an illustration, the proof net

\[
\begin{array}{c}
A \quad B \\
\hline
A \otimes B \\
\hline
( A \otimes B ) \otimes C \\
\hline
A^+ \quad B^+ \\
\hline
A^+ \otimes B^+ \\
\end{array}
\]

is the image under the map \( N : LLM \rightarrow PNO \) of both of the following proofs of \( \vdash ( A \otimes B ) \otimes C, \ A^+ \otimes B^+, \ C^+ \).

\[
\begin{array}{c}
\vdash A, \ A^+ \\
\hline
\vdash A \otimes B, \ A^+, \ B^+ \ \ (\otimes) \\
\vdash A \otimes B, \ A^+, \ B^+ \\
\hline
\vdash C, \ C^+ \ \ (\otimes) \\
\vdash ( A \otimes B ) \otimes C, \ A^+, \ B^+, \ C^+ \\
\hline
\vdash ( A \otimes B ) \otimes C, \ A^+ \otimes B^+, \ C^+ \ \ (\otimes)
\end{array}
\]
\[ \frac{\vdash A, A^+ \quad \vdash B, B^+}{\vdash A \otimes B, A^+, B^+} \quad (\otimes) \]
\[ \frac{\vdash A \otimes B, A^+, B^+}{\vdash A \otimes B, A^+ \otimes B^+} \quad (\otimes) \]
\[ \frac{\vdash A \otimes B, A^+ \otimes B^+}{\vdash (A \otimes B) \otimes C, A^+ \otimes B^+, C^+} \quad (\otimes) \]

To establish that every proof net in PNO is a correct proof structure, we must show that the map \( N: LLM \rightarrow PNO \) is surjective. This result requires a rather delicate argument.

**Theorem 6.0.8:** (Girard [1987])
The map \( N: LLM \rightarrow PNO \) is surjective.

**Proof:**
Let \( \beta \) be a proof net in PNO and let \( A_1, \ldots, A_n \) be a list of the terminal formula occurrences of \( \beta \). We prove, by induction on the number of links in \( \beta \), that there is a proof \( \pi \) in LLM of \( \vdash \sigma(A_1, \ldots, A_n) \), for some permutation \( \sigma \), such that \( \beta = N(\pi) \).

Suppose \( \beta \) has exactly one link. Then \( \beta \) must be of the form

\[
\begin{array}{c}
A \\
\hline
A^+
\end{array}
\]

in which case, take \( \pi \) to be the instance \( \vdash A, A^+ \) of the identity axiom scheme.

Suppose \( \beta \) has more than one link. Now \( \beta \) cannot consist of two or more non-connected axiom links since all trips in such proof structures are short.
So $\beta$ must contain either a CUT link or else a terminal $\otimes$ or $\&$ link.

Case (ii): $\beta$ has at least one terminal $\&$ link. We draw $\beta$ as follows:

Here, $\beta_0$ is the proof structure obtained from the proof net $\beta$ by removing the $\&$ link (so $A$ and $B$ are terminal in $\beta_0$). We claim that $\beta_0$ is a proof net. Arbitrarily select positions for all the switches in $\beta_0$ and suppose the additional $\&$ switch in $\beta$ is on "L". In $\beta$ we have a long trip of the form

$$A^\wedge, \ldots, B^\wedge, B^\wedge, \ldots, A^\wedge, A^\wedge, A^\wedge, B^\wedge, A^\wedge$$

This shows that the trip in $\beta_0$ of the form

$$A^\wedge, \ldots, B^\wedge, B^\wedge, B^\wedge, \ldots, A^\wedge, A^\wedge$$

is a long trip. Since the positioning of switches was arbitrary, $\beta_0$ is a proof net.
Since $\beta_0$ has one less link than $\beta$, the induction hypothesis gives us a proof $\Pi_0$ of $\vdash A, B, \Gamma$ (where $\Gamma$ is a sequence of all the terminals of $\beta_0$ other than $A$ and $B$) such that $\beta_0 = N(\Pi_0)$. Then we can take $\Pi$ to be the proof

$$
\begin{align*}
\text{\ldotp} & \\
\vdash A, B, \Gamma & \\
\vdash A \otimes B, \Gamma
\end{align*}
$$

(8)

And clearly, $\beta = N(\Pi)$.

Case (ii): $\beta$ has no terminal $\otimes$ links. Then $\beta$ must contain at least one CUT link or at least one terminal $\otimes$ link. Notice that CUT links and terminal $\otimes$ links are structurally similar: if we made a terminal $\otimes$ link out of premises $A$ and $A^\bot$, and stipulated that all tram trips are to run express (not stopping) through the tram stops $(A \otimes A^\bot)^v$ and $(A \otimes A^\bot)^w$ then the result would be the same as if we made a CUT link out of $A$ and $A^\bot$.

We say a terminal $\otimes$ link in a proof net $\beta$ splits if there are sub-proof nets $\beta_0$ and $\beta_1$ of $\beta$ such that every formula occurrence in $\beta$ other than the conclusion of the $\otimes$ link is in exactly one of $\beta_0$ and $\beta_1$, and the only link between $\beta_0$ and $\beta_1$ is the $\otimes$ link in question.
Likewise, we say a CUT link in a proof net $\beta$ splits if there are sub-proof nets $\beta_0$ and $\beta_1$ of $\beta$ such that every formula occurrence in $\beta$ is in exactly one of $\beta_0$ and $\beta_1$, and the only link between $\beta_0$ and $\beta_1$ is the CUT link in question.

\[ \begin{array}{c}
\beta_0 \\
A \\
\beta_1 \\
A^* \\
\hline
\text{CUT}
\end{array} \]

To complete the proof of the surjectivity of the map $N$, we need the following, by no means obvious, result.

**Theorem 6.0.9:** the Splitting Theorem (Girard [1987])

If $\beta$ is a proof net in $\mathbf{PNO}$ with more than one link but no terminal $\otimes$ links, then there is either a CUT link in $\beta$ which splits or else a terminal $\otimes$ link in $\beta$ which splits.

For now, we take the Splitting Theorem on faith. Suppose we have a CUT link in $\beta$, with premises $A$ and $A^*$, which splits. Applying the induction hypothesis to $\beta_0$ and $\beta_1$, we obtain proofs $\pi_0$ and $\pi_1$ in $\mathbf{LLM}$ of $\vdash A, \Gamma_0$ and $\vdash A^*, \Gamma_1$, respectively, such that $\beta_0 = N(\pi_0)$ and $\beta_1 = N(\pi_1)$. Then take $\pi$ to be the proof

\[
\begin{array}{c}
\Gamma_0 \\
\pi_0 \\
A, \Gamma_0 \\
\pi_1 \\
\hline
A^*, \Gamma_1 \\
\Gamma_0, \Gamma_1
\end{array} \quad (\text{cut})
\]

Clearly, $\beta = N(\pi)$. 
Similarly, if there is a terminal $\otimes$ link in $\beta$ which splits, then by applying the induction hypothesis to $\beta_0$ and $\beta_1$, we readily obtain the desired proof $\Pi$ in $\text{LLM}$ such that $N(\Pi) = \beta$.

**Observation 6.0.10:**

Let $\beta$ be a proof net in $\text{PNO}$ and fix an ordering, say $A_1, \ldots, A_n$, on the terminal formula occurrences of $\beta$. By the surjectivity of the map $N : \text{LLM} \to \text{PNO}$, there is a proof $\Pi_0$ in $\text{LLM}$ of $\Gamma \sigma(A_1, \ldots, A_n)$, for some permutation $\sigma$, such that $\beta = N(\Pi_0)$. Now apply the exchange rule to $\Pi_0$ to obtain a proof $\Pi$ of $\Gamma A_1, \ldots, A_n$. We can now define an object $\beta^* \in A_1^* \otimes \cdots \otimes A_n^*$ (identifying formulae with coherent spaces) by simply putting $\beta^* = \Pi^*$. To ensure that $\beta^*$ is well-defined, we have to verify that whenever $N(\Pi_0) = N(\Pi_1)$, $\Pi_1^* = \frac{1}{2} \tau(\xi) \ | \ z \in \Pi_0^* \tau$ for some permutation $\tau$. We omit the verification.

If it were the case that in any proof net without terminal $\&$ links, every CUT link and terminal $\otimes$ link splits, then the Splitting Theorem would be trivial. But this is not the case. For example, consider the following proof net.

```
  A^+ B^+  A B  C C^+  C^+ C
  A^+ \otimes B^+  (A \otimes B) \otimes (C \otimes C^+)
```

The central \( \otimes \) link splits but neither the \( \text{CUT} \) link nor the other \( \otimes \) link are splittable. Note also that the hypothesis 'Without terminal \( \otimes \) links' is necessary: look again at the proof net labelled (a) in Examples 6.0.1.

We will set out the main lines of argument in the proof of the Splitting Theorem, but omit some of the details.

**Lemma 6.0.11:**

Let \( \beta \) be a proof net, let \( \frac{A \otimes B}{A \otimes B} \) be a \( \otimes \) link in \( \beta \) and let \( t \) be any (necessarily long) trip in \( \beta \). If the given \( \otimes \) switch is on "L" then the trip \( t \) is of the form

\[
A^*, \ldots, A, A \otimes B, \ldots, A \otimes B^*, B^*, \ldots, B, A^*
\]

and if the given \( \otimes \) switch is on "R" then the trip \( t \) is of the form

\[
A^*, \ldots, A, B^*, \ldots, B, A \otimes B, \ldots, A \otimes B^*, A^*.
\]

Graphically, the trips are as follows.

[Diagram of proof nets with labeled switches]
Proof:
Suppose the \( \otimes \) switch is on "L". Assume, for a contradiction, that the next tram stop in the \( \otimes \) link to which the tram travels after \( A \otimes B \) is not \( A \otimes B^a \). The only other points of entry to the \( \otimes \) link are \( A \) and \( B \). We can eliminate \( A \), for otherwise we would have a short trip:

\[
A, A \otimes B, \ldots, A.
\]

So suppose the next stop in the \( \otimes \) link after \( A \otimes B \) is \( B \). The trip must be of the form:

\[
A, A \otimes B, \ldots, B, A^a, \ldots, A \otimes B^a, B^a, \ldots, A.
\]

By then when the \( \otimes \) switch is on "R" we can piece together a short trip:

\[
B, A \otimes B, \ldots, B.
\]

The argument is similar when we start with the \( \otimes \) switch on "R".

Using a similar sort of argument, we can also characterize the general pattern of trips through \( \& \) links.

Lemma 6.0.12:
Let \( \beta \) be a proof net, let \( \frac{A \otimes B}{A \otimes B} \) be a \( \& \) link in \( \beta \) and let \( t \) be any (long) trip in \( \beta \). If the given \( \& \) switch is on "L", then the trip \( t \) is of the form:

\[
A^a, \ldots, B, B^a, \ldots, A, A \otimes B, \ldots, A \otimes B^a, A^a
\]
and if the given switch is on "R" then the trip \( t \) is of the form

\[
A^\wedge, ..., B_v, A \otimes B_v, ..., A \otimes B^\wedge, B^\wedge, ..., A_v, A^\wedge.
\]

Graphically, the trips are as follows.

\[
\text{Switch on "L"}
\]

\[
\text{Switch on "R"}
\]

**Definition 6.0.13:**

Let \( \beta \) be a proof net in \( \mathbf{PNO} \), let \( A \) be any formula occurrence in \( \beta \) which is a premis of a link, and let \( t \) be any (long) trip in \( \beta \). The interval denoted \([A^\wedge, A_v]^t\) is the collection of all tram stops between \( A^\wedge \) and \( A_v \) on the trip \( t \), together with \( A^\wedge \) and \( A_v \). \([A^\wedge, A_v]^t\) is ordered in terms of the trip \( t \). When \( A \) is a formula occurrence in \( \beta \) which is the conclusion of a link, the interval \([A_v, A^\wedge]^t\) is defined similarly.

**Corollary 6.0.14:**

If \( \frac{A \odot B}{A \otimes B} \) is any \( \otimes \) link in a proof net \( \beta \), then for each trip \( t \) in \( \beta \), the intervals \([A^\wedge, A_v]^t\), \([B^\wedge, B_v]^t\) and \([A \otimes B_v, A \otimes B^\wedge]^t\) are pair-wise disjoint and their union is the collection of all tram stops in \( \beta \).
Corollary 6.0.15:
If \( \frac{A}{\text{cut}} A^* \) is any cut link in a proof net \( \beta \),
then for each trip \( t \) in \( \beta \),
\( [A^*, A]_t \cap [A^+, A^+_v]_t = \emptyset \),
and \( [A^*, A]_t \cup [A^+, A^+_v]_t \) contains every transport
in \( \beta \).

Definition 6.0.16:
Let \( \beta \) be a proof net and let \( \Lambda \) be either a premis of
a \( \otimes \) link in \( \beta \) or a premis of a cut link in \( \beta \).
We define
\[
(A^*, A^+_v)_t \overset{\triangleq}{=} \{ C \mid C^* \in [A^*, A]_t \text{ and } C \in [A^+, A^+_v]_t \}
\]
that is, the collection of all formula occurrences in \( \beta \) both
of whose transports are in the interval \( [A^*, A^+_v]_t \).
We define the empire of \( \Lambda \), denoted \( e\Lambda \), as follows:
\[
e\Lambda \overset{\triangleq}{=} \{ C \mid \text{for all trips } t \text{ in } \beta, \ C \in (A^*, A^+_v)_t \}
\]
So \( e\Lambda \) is the intersection of all the \( (A^*, A^+_v)_t \); trivially,
\( \Lambda \in e\Lambda \).

Corollary 6.0.17:
If \( \frac{A}{\otimes B} A^* \) is any \( \otimes \) link in a proof net \( \beta \),
then \( e\Lambda \cap eB = \emptyset \).

Corollary 6.0.18:
If \( \frac{A}{\text{cut}} A^* \) is any cut link in a proof net \( \beta \),
then \( e\Lambda \cap eA^* = \emptyset \).
Lemma 6.0.18: (Girard [1987])

Let $\frac{A \quad B}{A \otimes B}$ be any $\otimes$ link in a proof net $\beta$.

(i) If $C$ is linked to $C^\perp$ in $\beta$ by an axiom link $\frac{C}{C^\perp}$ then $C \in eA$ iff $C^\perp \in eA$.

(ii) If $C$ is linked to $C^\perp$ in $\beta$ by a CUT link $\frac{C \quad C^\perp}{\text{CUT}}$ then $C \in eA$ iff $C^\perp \in eA$.

(iii) If $\frac{C \quad D}{A \otimes B}$ is any $\otimes$ link in $\beta$ distinct from $\frac{A \quad B}{A \otimes B}$ then $C \in eA$ iff $C \otimes D \in eA$, and $D \in eA$ iff $C \otimes D \in eA$.

(iv) If $\frac{C \quad D}{C \otimes D}$ is any $\&$ link in $\beta$ then $C \in eA$ and $D \in eA$ iff $C \otimes D \in eA$.

(v) If $C$ is an hereditary premise of $A$ in $\beta$ then $C \in eA$.

Proof:

(v) is a consequence of (i) - (iv).

(i) and (ii) are immediate: in the first case we have $t(C^\&v) = t(C^\perp) + 1$ and $t(C^\perp) = t(C^\&v) + 1$ for all trips $t$ in $\beta$; in the second case we have $t(C^\&v) = t(C^\perp) + 1$ and $t(C^\perp) = t(C^\&v) + 1$ for all trips $t$ in $\beta$.

The proofs of (iii) and (iv) involve teasing out the consequences of the general pattern of trips through $\otimes$ and $\&$ links (Lemmas 6.0.11 and 6.0.12). As an illustration, we give the proof of the second part of (iii): if $C \otimes D \in eA$ then $C \in eA$ and $D \in eA$.

If $C \otimes D \in eA$ then for all trips $t$ in $\beta$, $C \otimes D^\& \in [A^\&_v A^\&_v]_t$ and $C \otimes D^\& \in [A^\&_v A^\&_v]_t$. Let $t$ be a trip in $\beta$ such
that the \( \frac{C}{D}_{\text{COD}} \) switch is on "L".

Then \( C \in [A^v, Av]^t \) and \( D^v \in [A^v, Av]^t \). Now suppose
that \( Dv \notin [B^v, Bv]^t \). Then either \( Dv \in [B^v, Bv]^t \) or else
\( Dv \in [A \odot Bv, A \odot B^v]^t \). Assume the former and consider the
sequence of tramstops \( B^v, \ldots, Dv \) on the trip \( t \). By changing
the \( C \odot D \) switch to "R" and setting the \( A \odot B \) switch
on "R" also, we can piece together a short trip \( t' \) of
the form

\[ Av, B^v, \ldots, Dv, C \odot Dv, \ldots, Av. \]

Hence \( Dv \notin [B^v, Bv]^t \). For similar reasons, \( Dv \notin [A \odot Bv, A \odot B^v]^t \).
So we must have \( Dv \in [A^v, Av]^t \) for all trips \( t \) in \( \beta \) with
the \( C \odot D \) switch on "L", and for all such trips \( t \),
\( t(C^v) = t(Dv) + 1 \). Hence \( D \in (A^v, Av)^t \) and \( C \in (A^v, Av)^t \)
for all trips \( t \) with the \( C \odot D \) switch on "L".

For trips in \( \beta \) with the \( C \odot D \) switch on "R", repeat the
above argument with \( C \) and \( D \) exchanged.

\[ \square \]

\textbf{Corollary 6.0.19:}

Let \( \frac{A}{A^+}_{\text{cut}} \) be any \textbf{CUT} link in a proof net \( \beta \).

(i) If \( C \) is linked to \( C^v \) in \( \beta \) by an axiom link \( \frac{C}{C^v} \)
then \( C \in eA \iff C^v \in eA \).
(ii) If \( \frac{c}{cut} \) is a \( \textsc{cut} \) link in \( \beta \) distinct from \( \frac{A}{cut} \),
then \( c \in eA \) iff \( c^1 \in eA \).

(iii) If \( \frac{c \cdot d}{c \odot b} \) is a \( \odot \) link in \( \beta \),
then \( c \in eA \) iff \( c \odot d \in eA \), and
\( d \in eA \) iff \( c \odot d \in eA \).

(iv) If \( \frac{c \cdot d}{c \odot b} \) is a \( \& \) link in \( \beta \),
then \( c \in eA \) and \( d \in eA \) iff \( c \odot d \in eA \).

(v) If \( C \) is an hereditary premis of \( A \) in \( \beta \),
then \( c \in eA \).

Let \( \frac{A \cdot B}{A \odot B} \) be a terminal \( \odot \) link in a proof net \( \beta \). The
empires \( eA \) and \( eB \) are our candidates for the under-
lying sets of disjoint sub-proof nets \( \beta_0 \) and \( \beta_1 \) such
that \( \beta \) is split as follows:

\[
\begin{array}{c}
\begin{array}{c}
\beta_0 \\
A
\end{array} \\
\begin{array}{c}
\begin{array}{c}
\beta_1 \\
B
\end{array}
\end{array}
\end{array}
\]
\( A \odot B \)

Now if \( eA \) contains any part (premis or conclusion) of
either an axiom link, a \( \textsc{cut} \) link or a \( \odot \) link other
than \( \frac{A \cdot B}{A \odot B} \), then all the formula occurrences in that
link are in \( eA \). The only problem cases are \( \& \) links
in \( \beta \) where one premis is in \( eA \) and the other premis
is in \( eB \) so the conclusion is in neither.

**Theorem 6.0.20:** The Trip Theorem (Girard [1987]).
Let \( A \) be a premis of either a \( \textsc{cut} \) link or a \( \odot \) link in a
proof net \( \beta \). Then there is a trip \( t \) in \( \beta \) such that
\( eA = (A^\perp, A_\downarrow)_t \).
Proof:
Recall that for all trips \( t \) in \( \beta \), \( eA \subseteq (A^*, A\wedge)\). The theorem asserts the existence of a trip which realizes \( eA \).

The trip \( t \) in \( \beta \) is defined as follows. For each \( S \) link \( \frac{C}{D} \) such that exactly one premise is in \( eA \) (so \( C \bar{\in} eA \)), if \( C \in eA \) then set that \( S \) switch to "R", and if \( D \in eA \) then set that \( S \) switch to "L".

\[
\begin{align*}
\text{"R":} & \quad \frac{C}{D} \quad \text{"L":} & \quad \frac{C}{D} \\
& \quad C \bar{\in} eA & \quad C \in eA
\end{align*}
\]

All other switches in \( \beta \) are set arbitrarily.

Now suppose \( E \in (A^*, A\wedge) \) but \( E \notin eA \). Since \( A \in eA \), our supposition means that at some stage after \( A^* \) in the interval \( [A^*, A\wedge] \), we travel from an antecedent of a formula \( F \in eA \) to, say, \( E \). So either \( t(E) = t(F^*) + 1 \) or \( t(E) = t(F\wedge) + 1 \). What can the link between \( E \) and \( F \) be? \( F \) cannot be \( E^* \), linked to \( E \) by either an axion link or a \( \text{CUT} \) link, because in both cases we have \( E^* \in eA \) iff \( E \in eA \). Likewise, \( F \) cannot be any part of a \( \otimes \) link of which \( E \) is a part, and \( F \) cannot be the conclusion of a \( S \) link of which \( E \) is a premise. The only remaining possibility is that \( F \) is a premise of a \( S \) link of which \( E \) is the conclusion, say \( E \) is \( F \otimes G \). But then \( G \notin eA \) for otherwise \( E \in eA \). So we have exactly one premise of a \( S \) link in \( eA \). The trip \( t \) is defined so that after \( F \), we turn back to \( F^* \), never getting to the conclusion \( F \otimes G = E \). Hence \( E \notin (A^*, A\wedge) \).
Definition 6.0.20:
If $\beta$ is any proof net then $1|\beta|$ is the collection of all formula occurrences in $\beta$.

Corollary 6.0.21:
Let $\frac{A}{\text{cut}} \frac{A}{A \otimes B}$ be any terminal $\otimes$ links in a proof net $\beta$.

$1|\beta| = eA \cup eB \cup A \otimes B \iff$ there is no $\otimes$ link $\frac{C}{C \otimes D}$ in $\beta$ such that either $C \in eA$ and $D \in eB$ or $C \in eB$ and $D \in eA$.

Proof:
Suppose there is a $\otimes$ link $\frac{C}{C \otimes D}$ in $\beta$ with one premise in each of $eA$ and $eB$. Then $C \otimes D \notin eA$ and $C \otimes D \notin eB$, hence $1|\beta| \neq eA \cup eB \cup A \otimes B \otimes B \otimes B$.

Conversely, suppose there are no $\otimes$ links in $\beta$ with one premise in each of $eA$ and $eB$. Then there are no constraints on $\otimes$ links satisfying the Trip Theorem so for all trips $t$ in $\beta$, $eA = (A^\otimes, A^\otimes)_t$ and $eB = (B^\otimes, B^\otimes)_t$.

The remaining two transit stops in $\beta$ are $A \otimes B^\otimes$ and $A \otimes B^\otimes$, hence $1|\beta| = eA \cup eB \cup A \otimes B \otimes B$.

Corollary 6.0.22:
Let $\frac{A}{\text{cut}} \frac{A}{A^\otimes}$ be any cut link in a proof net $\beta$.

$1|\beta| = eA \cup eA^\otimes \iff$ there are no $\otimes$ links in $\beta$ with one premise in each of $eA$ and $eA^\otimes$.

Note that terminal $\otimes$ links can give rise to the 'one premise each way' phenomenon (see Examples 6.0.1 (a)) but not all instances of the phenomenon come from terminal $\otimes$ links.
Lemma 6.0.22:
If \( \frac{A}{B} \) is a terminal \( \otimes \) link in a proof net \( \beta \) such that \( \mid \beta \mid = eA \cup eB \cup \frac{A}{B} \), then the proof structures \( \beta_0 \) and \( \beta_1 \) obtained from \( \beta \) by restriction to \( eA \) and \( eB \) respectively are proof nets.

Proof:
The hypothesis implies that for each trip \( t \) in \( \beta \), \( eA = (A^v, A v)_t \) and \( eB = (B^v, B v)_t \). Moreover, for all formula occurrences \( C \in \mid \beta \mid \), \( eA \in [A^v, A v]_t \) iff \( C^v \in [A^v, A v]_t \), and likewise for \( [B^v, B v]_t \). Let \( t \) be any trip in \( \beta_0 \) starting at \( A^v \). Then \( t \) must follow the course of an interval \( [A^v, A v]_t \), for some trip \( t \) in \( \beta \), and then loop back to \( A^v \) (since \( A \) is terminal in \( eA \)). Hence \( t \) is a long trip. Likewise, all trips \( t \) in \( \beta_1 \) are long trips.

Corollary 6.0.23:
If \( \frac{A}{A^v} \) is a CUT link in a proof net \( \beta \) such that \( \mid \beta \mid = eA \cup eA^v \), then the proof structures \( \beta_0 \) and \( \beta_1 \) obtained from \( \beta \) by restriction to \( eA \) and \( eA^v \) respectively are proof nets.

Now we are almost ready to prove the Splitting Theorem; we need one more preparatory lemma.
Lemma 6.0.24: (Girard [1987])

Let \( A \) be a premise of either a \( \otimes \) link or a \( \text{cut} \) link in a proof net \( \beta \).

If \( \frac{c}{a} \otimes b \) is a \( \otimes \) link in \( \beta \) such that \( c \otimes d \) is an hereditary premise of \( A \)
then \( e_c u e_d \leq e_A \).

Proof:

Let \( t \) be a trip in \( \beta \) such that \( e_A = (A^*, A) t \) and the \( \otimes \) switch is on "L". Then the interval \([A^*, A] t\) is of the form

\[ A^*, \ldots, \otimes d^*, d^*, \ldots, d^*, c^*, \ldots, c^*, \otimes d, \ldots, A \]

Hence \( e_d c \leq (d^*, d^*) t \leq (A^*, A) t = e_A \) and likewise, \( e_c \leq e_A \).

Proof of theorem 6.0.9 (The Splitting Theorem)

Let \( \beta \) be a proof net with more than one link but no terminal \( \& \) links. Survey the collection of all \( \otimes \) links \( \frac{A}{A \otimes B} \) and \( \text{cut} \) links \( \frac{A^*}{\text{cut}} \) in \( \beta \), and choose one such that \( e_A u e_H \) is maximal with respect to inclusion, where \( H \) is either \( B \) or \( A^* \) depending on the case.

If a \( \otimes \) link \( \frac{A}{A \otimes B} \) has been chosen, then we claim \( A \otimes B \) is terminal in \( \beta \). Suppose otherwise. Then there must be either a terminal \( \otimes \) link \( \frac{c}{a} \otimes b \) or else a \( \text{cut} \) link \( \frac{c}{\text{cut}} \) such that \( A \otimes B \) is an hereditary premise of, say, \( C \). By Lemma 6.0.24, we have \( e_A u e_B \leq e_C \), hence \( e_A u e_B \leq e_C u e_D \) or \( e_A u e_B \leq e_C u e_C \), contradicting the assumption of maximality.
If a \text{CUT} link, \( \frac{A}{\text{cut}} \), has been chosen, then it is necessarily a terminal link of \( \beta \).

Assume a \( \otimes \) link has been chosen, the argument for a \text{CUT} link with \( eA \vee eA^* \) maximal can be extracted from what follows.

Now suppose that \( eA \vee eB \vee A \otimes B \beta \neq 1 \beta \). Then, by Lemma 6.0.21, there is a \( \otimes \) link, \( \frac{C \otimes D}{C \otimes D} \), such that either \( C \in eA \) and \( D \in eB \) or else \( C \in eB \) and \( D \in eA \). Now \( C \otimes D \) must be an hereditary premise of one of the premises of either a \text{CUT} link or a terminal \( \otimes \) link.

For definiteness, let \( C \otimes D \) be an hereditary premise of \( F \) in \( \frac{F}{\text{cut}} \). By Corollary 6.0.19, we have \( C \otimes D \in eF \), \( C \in eF \) and \( D \in eF \). So this \( \otimes \) link puts no constraint on trips satisfying the Trip Theorem with respect to \( eF \).

Suppose \( C \in eA \) and \( D \in eB \). Now we can choose a trip \( t \) in \( \beta \) such that \( eF=(F^*, F_v)_t \) and \( eB=(B^*, B_v)_t \); in particular, the \( C \otimes D \) switch is on "L". By Lemma 6.0.12, the interval \([F^*, F_v)_t\) is of the form

\[ F^*, \ldots, C \otimes D^*, C^*, \ldots, D_v^*, D^*, \ldots, C_v, C \otimes D_v, \ldots, F_v. \]

The we must have \( B^* \) between \( C^* \) and \( D_v \), and \( B_v \) between \( D^* \) and \( C_v \). Hence \( eB=(B^*, B_v)_t \leq (F^*, F_v)_t = eF \).

By interchanging \( C \) and \( D \) we get \( eA \subseteq eF \).
Hence \( eA \vee eB \leq eF \vee eF^* \), contradicting maximality.
So we must have $e^A + e^B \triangleq e^{A \otimes B} = 1_{\beta}$. And by Lemma 6.0.22, the $\otimes$ link $\frac{A}{A \otimes B}$ splits $\beta$, as required.

6.1 Normalization in PNO

Normalization in PNO is quite quick and painless.

**Definition 6.1.0:**
Let $\beta$ be a proof net in PNO.
We say $\beta$ is **normal** iff $\beta$ contains no CUT links.
$\beta$ **converts** to a proof structure $\beta'$, written $\beta \cong \beta'$, when one of the following cases holds:

(i) **axiom conversion (AC):**
$\beta'$ is obtained from $\beta$ by replacing a configuration of the form

\[
\begin{array}{cccc}
\vdots & \text{CUT} & \vdots & \vdots \\
\end{array}
\]

by

\[
\begin{array}{cccc}
A & A^* \\
\vdots & \vdots \\
\end{array}
\]

(ii) **multiplicative symmetric conversion ($\otimes/\&-Sc$):**
$\beta'$ is obtained from $\beta$ by replacing a configuration of the form

\[
\begin{array}{cccc}
A & B & A^* & B^* \\
\vdots & \vdots & \vdots & \vdots \\
\end{array}
\]

by

\[
\begin{array}{cccc}
A & A^* & B & B^* \\
\vdots & \text{CUT} & \text{CUT} & \vdots \\
\end{array}
\]
Observe that for any proof net $\beta$, there is a $\beta'$ such that $\beta \cong \beta'$ iff $\beta$ contains at least one CUT link.

The axiom conversion scheme (AC) may appear too restrictive; we seem to have omitted from consideration configurations of the form

$$
\begin{array}{c}
\vdots \\
A & A' & A \\
\vdots \\
\text{CUT}
\end{array}
$$

But recall that axiom links can only have literals as their conclusions (Definition 6.0.0). So whenever we have a CUT against the conclusion of an Axiom link, the other premise of the CUT link must also be a literal.

We can also ignore configurations of the form

$$
\begin{array}{c}
\vdots \\
A & A' \\
\text{CUT}
\end{array}
$$

Since such proof structures are not proof nets ((c) in Examples 6.0.4).

**Definition 6.1.1:**

Let $\beta$ be a proof net in \textit{PNO} containing $p$ formula occurrences. The size of $\beta$, $\mathcal{S}(\beta)$ = $p$.

We have yet to verify that when $\beta$ is a proof net and $\beta \cong \beta'$ then $\beta'$ is a proof net.
Lemma 6.1.2:
If \( \beta \) is a proof net in \( \text{PNO} \) and \( \beta \) con \( \beta' \), then \( \beta' \) is a proof net with the same terminal formula occurrences as \( \beta \) and \( s(\beta') < s(\beta) \).

Proof:
(i) Suppose \( \beta \) con \( \beta' \) by the axiom conversion scheme (Ac).
Each (long) trip in \( \beta \) is of the form
\[ A^\wedge, A^\vee, A^\wedge, A^\vee, \ldots, A^{++}, A^\vee, A^{++}, A^\vee, \ldots, A^\wedge \]
where \( A \) (respectively, \( A^\wedge \)) denotes the occurrence of the formula \( A \) (\( A^\wedge \)) which is a premis of the CUT link under examination. By identifying the two occurrences of \( A \) and the two occurrences of \( A^\wedge \), we obtain a long trip in \( \beta' \) of the form
\[ A^\wedge, A^\vee, \ldots, A^{++}, A^\vee, \ldots, A^\wedge \]
but every trip in \( \beta' \) is of this form, so \( \beta' \) is a proof net.
If \( s(\beta) = p \) then \( s(\beta') = p - 2 \).

(ii) Suppose \( \beta \) con \( \beta' \) by the scheme (\( \otimes/\& - \text{sc} \)).
Arbitrarily select positions for all the switches in \( \beta' \). This induces a switch configuration for all the switches in \( \beta \) except for the switches for the \( \otimes \) link and the \( \& \) link removed by this conversion. Suppose both those switches are on "R". (the argument is similar for the other three combinations.) By Lemmas 6.0.11 and 6.0.12, the trip \( t \) in \( \beta \) starting at \( A^\wedge \) is of the form
\[ A^\wedge, \ldots, A^\vee, B^\wedge, \ldots, B^\vee, A \otimes B, (A^\wedge \& B^\wedge)^\wedge, B^\wedge, \ldots, A^\wedge, A^\vee, A^{++}, \ldots, B^\vee, (A^\wedge \& B^\wedge)^\vee, A \otimes B^\wedge, A^\wedge. \]
Hence the trip $t'$ in $β'$ determined by the given configuration of switches in $β'$ must be of the form

$$A^+, \ldots, A_v, A^+, \ldots, \ldots, B^+, B^+, \ldots, B_v, B^+, \ldots, A^+, A^+$$

Since $t$ is a long trip, so is $t'$. Hence $β'$ is a proof net. If $s(β) = p$ then $s(β') = p - 2$.

**Definition 6.1.3:**

Let $β$ and $β'$ be proof nets. We say $β$ reduces to $β'$, denoted $β \rightarrow β'$, if there is a sequence of conversions

$$β = β_0 \text{ con } β_1 \text{ con } \ldots \text{ con } β_n, \text{ con } β_n = β'$$

from $β$ to $β'$. (So the relation $\rightarrow$ is the transitive closure of con.) Such a sequence of conversions is called a reduction sequence from $β$ to $β'$ of length $n$ ($n > 0$; $n = 0$ when $β = β'$). We use the notation $β \rightarrow^n β'$ to indicate the existence of a reduction sequence from $β$ to $β'$ of length $n$. 
A proof net $\beta$ is **weakly normalizable** iff for some $n < \omega$ and some $\beta'$ which is normal, $\beta \leadsto_n \beta'$. In this case, we say $\beta'$ is a **normal form** for $\beta$.

For each proof net $\beta$, define

$$L(\beta) \triangleq \sup \{ n \mid \exists \beta' (\beta \leadsto_n \beta') \}.$$

A proof net $\beta$ is **strongly normalizable** iff $L(\beta) < \omega$.

If every reduction sequence starting from a given proof net $\beta$ is of bounded length, then every reduction sequence starting from $\beta$ can be extended to one whose final term is normal. And conversely.

A proof net $\beta$ may fail to be strongly normalizable if there is a sequence of conversions starting from $\beta$ such that the conversions do not decrease the size of the proof nets involved; in particular, if there is a 'loop' $\beta \leadsto \beta' \leadsto \beta'' \leadsto \beta' \leadsto \beta'' \leadsto \ldots$ where $\beta'$ and $\beta''$ are not normal. Such things do not happen in PNO.

**Proposition 6.1.4:** (Girard [1981])

The relation $\leadsto$ on PNO has the Church-Rosser property. That is, if $\beta \leadsto \beta_0$ and $\beta \leadsto \beta_1$, then there is some $\beta'$ such that $\beta_0 \leadsto \beta'$ and $\beta_1 \leadsto \beta'$.

We omit the proof; it is fairly straightforward but messy. If one contemplates for a moment the conversion schemes (AC) and ($\otimes$/$\otimes$-SC) then one readily forms the belief that the order of any two conversions can be
Swapped without altering the final result.

**Theorem 6.1.5:** (Girard [1987])

Every proof net in PNO is strongly normalizable and has a unique normal form.

**Proof:**

Let \( \beta \) be a proof net in PNO and suppose \( s(\beta) = p \).

Suppose \( \beta \leadsto \beta' \). By Lemma 6.1.2, \( n < s(\beta) \). Hence \( L(\beta) < s(\beta) \). If \( \beta' \) and \( \beta'' \) are two normal forms for \( \beta \) then by Proposition 6.1.4, \( \beta' = \beta'' \).

In the last part of Section 5 we announced the result that if \( \beta \leadsto \beta_0 \) then \( \beta^* = \beta_0^* \). We could verify this for PNO now: we would have to find a proof \( \Pi \) in LLM such that \( \beta = N(\Pi) \), mimic the conversions in LLM to obtain a \( T_0 \) such that \( \beta_0 = N(T_0) \) and then check that \( \Pi^* = T_0^* \) (cf. the discussion following Observation 5.3.1).

We could, but we won't. The normalization of proof nets is a 'bottom-up' procedure: we push cuts upwards until we get to axiom links at the top, then get rid of the cuts altogether. Inductively defining \( \Pi^* \) for a proof \( \Pi \), and derivatively, \( \beta^* \) for a proof net \( \beta \), is a 'top-down' process. In Section 6.3, after we have developed the full system of proof nets PNI!, adequate for LL!, we shall present Girard's alternative 'bottom-up' characterization of \( \beta^* \). Then in Section 6.4, where we set out the normalization procedure for PNI!, we shall confirm that when \( \beta \leadsto \beta_0 \) then \( \beta^* = \beta_0^* \).
6.2 Axiom boxes and PNT

Our task is to extend PNO so as to obtain systems of proof nets adequate for LL and LL!. Now a cheap way of extending a sequent calculus or a natural deduction system is by adding new axiom schemes. Can we do that with proof nets?

**Definition 6.2.0:**
A **axiom box** is a link of the form

\[ A_1 - A_2 - \ldots - A_n \]

where \( A_1, \ldots, A_n \) are formulae in the language \( \mathcal{L}_1 \) of modal linear logic, \( n \geq 1 \). \( A_1, \ldots, A_n \) are the conclusions of such a link; there are no premises.

A **proof⁺ structure** is an object consisting of occurrences of formulae of \( \mathcal{L}_1 \) and links between these occurrences of formulae. In addition to axiom links, cut links, \( \otimes \) links and \( \& \) links, we also allow axiom boxes to link formula occurrences. With each axiom box with \( n \) conclusions, \( n \geq 1 \), we associate a switch with \((n-1)!\) positions. Each position of the switch corresponds to an \( n \)-cycle \( \sigma \) in the permutation group \( S_n \). When the switch is set at \( \sigma \), from trips \( \tau \) satisfy the following condition:

\[ \text{for } i = 1, \ldots, n, \quad t(A_{\sigma(i)}^\tau) = t(A_i^\tau) + 1 \]

where \( A_1, \ldots, A_n \) are the conclusions of the given axiom box.

A **proof⁺ net** is a proof⁺ structure which admits no short trips. Let \( \text{PNO}^+ \) denote the collection of all proof⁺ nets.
Examples 6.2.1:

(i) Axiom links \( A \rightarrow A \) are axiom boxes; the switch has one position.

(ii) The switch associated with an axiom box with three conclusions has two positions: the 3-cycles \((123)\) and \((132)\). When \(\sigma = (123)\), long trips are as follows.

![Diagram](image)

When \(\sigma = (132)\), long trips are as follows.

![Diagram](image)

(iii) We are particularly interested in the axiom box with one conclusion

\[
\begin{array}{c}
1
\end{array}
\]

and a switch with \(0! = 1\) position, and axiom boxes of the form

\[
\begin{array}{c}
T \rightarrow A_1 \rightarrow \cdots \rightarrow A_n
\end{array}
\]

which have a switch with \(n!\) positions.
In our definition of proof^+ structures, we allow any formula of $L_i$ to be a conclusion of an axiom box. So corresponding to PNO^+, we have the sequent calculus LLM^+ which is obtained from LLM by adjoining infinitely many axioms, one for each sequence of formulae of $L_i$.

**Theorem 6.2.2:** (Girard [1987])

If $\Gamma$ is a proof of $\vdash A_i, \ldots, A_n$ in LLM^+

then we can naturally associate with $\Gamma$ a proof^+ net $N^+(\Gamma)$ in PNO^+ whose terminal formula occurrences are exactly $A_i, \ldots, A_n$.

**Proof:**

Modify the proof of Theorem 6.0.6 in the obvious way.

**Theorem 6.2.3:** (Girard [1987])

The map $N^+: LLM^+ \rightarrow PNO^+$ is surjective.

**Proof (sketch):**

We have to show that the empires eA retain their nice properties (Lemma 6.0.18 and Corollary 6.0.19) when axiom boxes are admitted. In particular, we have to ensure that when $A$ is a premise of either a CUT link or a $\otimes$ link in a proof net $\beta$ in PNO^+, and $C_i$ and $C_j$ are two conclusions of an axiom box

$$\boxed{\ldots \quad C_i \quad \ldots \quad C_n}$$

in $\beta$, then $C_i \in eA$ iff $C_j \in eA$. This follows from the fact that any long trip $t$ in $\beta$ can be expressed
as the disjoint union of the intervals \([C_i^*, C_i]\) for \(i=1, \ldots, n\). (Look again at (ii) in Examples 6.2.1.) Other than this, the rest of the results leading up and including the Splitting Theorem go through almost unchanged.

\[\]

Of course, the system PNO+ (and LLM+) is not terribly interesting; its purpose is only to record the fact that axiom boxes work.

If one scrutinizes the inference rules of the sequent calculus LL!, then, after a few moments reflection, one will notice that the rules (Fr), (sn0), (D?) and (C?) are easily mimicked as links. This is because these rules, like (⊗), (⊗) and (cut), place no restriction on the parametric sequences (or 'contexts', as Girard calls them) occurring in the premises sequents, and whatever parametric sequences there are reappear without change in the conclusion sequent. Now we have just established that the axioms and axiom schemes of LL! can be mimicked as axiom boxes. So this leaves four inference rules to be accounted for. They pose two distinct sorts of problems:

I. the 'nothing to link on to' problem:
   - the rules (I) and (Th?); and

II. the 'global constraint' problem:
   - the rules (&) and (!).
Girard deals with the four rules uniformly with a device called a **proof box**. A proof box is like an axiom box except that it has one or more proof nets inside it. Proof boxes are a source of complication when it comes to the normalization procedure. As Girard puts it, they are ‘...moments where we restore the sequent (i.e. the sequential !) structure. Their use is therefore a bridle to parallelism.’ (Girard [1987], p.43) Proof boxes also make a mess of \( \lambda \)m networks.

Girard's strategy for dealing with the rule (\( \perp \)) is as follows. From a proof net \( \beta \) with terminals \( C_1, ..., C_n \), form a proof net \( \beta' \) with terminals \( \perp, C_1, ..., C_n \) as illustrated:

![Proof Box Diagram](image)

To mimic the rule (\( \text{Th} \)), replace \( \perp \) with \( \text{?A} \) where A is any formula of \( \mathcal{L} \):

The problem with the rules (\( \perp \)) and (\( \text{Th} \)) is that they 'create' a new formula out of nothing (which is just to say, they involve **thinning**) so there is no premis formula to 'link' on to. But with regard to parametric sequences, they make no demands and are perfectly conservative. We cannot 'tack on' a \( \perp \) or a \( \text{?A} \) at the bottom of a proof net (or structure) but we **can** use axiom boxes so that all our thinning business is done at the top. In what follows, we part company...
With Girard and abandon the use of proof boxes to mimick the rules (1) and (th?). (Girard [1987], p. 41, does remark that the proof boxes for thinning "...seem of very limited interest.")

**Definition 6.2.4:**

A new proof structure is an object consisting of occurrences of formulae of $L_1$ and links between these occurrences of formulae. The links are of the following kinds:

1. **axiom boxes:**
   1.1. generalized axiom link: $k+m+2$ conclusions; $k \geq 0, m \geq 0$.

   $$\vdash \cdots \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdash \vdast
(2) **unary links:**

(2.1) $\text{FST} \oplus \text{link}:
\[
\begin{array}{c}
A \\
A \oplus B
\end{array}
\]

(2.2) $\text{SNQ} \oplus \text{link}:
\[
\begin{array}{c}
B \\
A \oplus B
\end{array}
\]

(2.3) $\text{DQ} \oplus \text{link}:
\[
\begin{array}{c}
A \\
?A
\end{array}
\]

(3) **binary links:**

(3.1) $\text{CUT} \oplus \text{link}:
\[
\begin{array}{c}
A \\
A^* \\
\text{CUT}
\end{array}
\]

(3.2) $\text{\&} \oplus \text{link}:
\[
\begin{array}{c}
A \\
B \\
A \& B
\end{array}
\]

(3.3) $\text{\&} \oplus \text{link}:
\[
\begin{array}{c}
A \\
B \\
A \& \neg B
\end{array}
\]

(3.4) $\text{CQ} \oplus \text{link}:
\[
\begin{array}{c}
?A \\
?A
\end{array}
\]

We also require that:

(i) every occurrence of a formula in a new proof structure is the conclusion of exactly one link; and

(ii) every occurrence of a formula in a new proof structure is the premise of at most one link.

**Definition 6.2.5:**

*Tram trips* in a new proof structure are defined component-wise.

(1) The scheduling of trips through axiom boxes is as specified in Definition 6.2.0.
(2) Tram tracks in the vicinity of a unary link are as follows.

\[ \begin{array}{c}
A \\
C \\
\end{array} \]

The tracks are such that every trip \( t \) satisfies

(i) \( t(A^c) = t(C^c) + 1 \), and

(ii) \( t(Cv) = t(Av) + 1 \).

(3) The scheduling of trips through cut links, \( \otimes \) links

and \( \& \) links is as specified in Definition 6.0.3.

The scheduling of trips through \( \wedge \) links follows

the same pattern as for \( \& \) links.

A new proof structure is a new proof net if it does not admit any short trips. Let \( \text{PNT} \) denote the collection of all new proof nets.

Now that we have got through the definitions, we can drop the modifier 'new'; until further notice, 'proof net' will mean an element of \( \text{PNT} \). (The 'T' is for 'temporary' and also for 'tams'.) Let \( \text{LLT} \) denote

the (admittedly unnatural) subsystem of \( \text{LL}! \) obtained from \( \text{LL}! \) by omitting the rules (3) and (1).

Theorem 6.2.6:

If \( \Pi \) is a proof of \( \vdash A_1, \ldots, A_n \) in \( \text{LLT} \)

then we can naturally associate with \( \Pi \) a proof net

\( N(\Pi) \) in \( \text{PNT} \) whose terminal formula occurrences

are exactly \( A_1, \ldots, A_n \).
Proof:
The only cases involving any novelty are those of the
rules (1) and (Th?).

(1) Suppose \( \Pi \) is obtained from \( \Pi_0 \) by the rule (1):
\[
\frac{\Gamma \vdash B_1, \ldots, B_n}{\vdash \bot, B_1, \ldots, B_n} \quad (1)
\]

Assume we have a proof net \( \beta_0 = N(\Pi_0) \) with terminals
\( B_1, \ldots, B_n \). We construct a proof structure \( \beta \) from \( \beta_0 \) as
follows.

If \( \beta_0 \) has a generalized axiom link
\[
\begin{array}{c}
\bot \vdash ? \Gamma \vdash C \vdash C^+ \\
\end{array}
\]
then replace that link with
\[
\begin{array}{c}
\bot \vdash \bot \vdash ? \Gamma \vdash C \vdash C^+ \\
\end{array}
\]

If \( \beta_0 \) has more than one generalized axiom link, choose
the one that contains an hereditary premise of \( B_i \) for
the least \( i \) (with respect to the ordering \( B_1, \ldots, B_n \)).

Otherwise, if \( \beta_0 \) has a \( \bot \) axiom box
\[
\begin{array}{c}
\bot \vdash ? \Gamma \\
\end{array}
\]
then replace that box with the generalized axiom link
\[
\begin{array}{c}
? \Gamma \vdash \bot \vdash \bot \\
\end{array}
\]
(This is why we have atomic formulae of \( \bot \) rather than
literals in the 'old axiom link' position.) If there
is more than one \( \bot \) axiom box, choose as above.
Otherwise \( \beta_0 \) must contain a \( T \) axiom box, say
\[
\begin{array}{c}
T \vdash ? X \\
\end{array}
\]
If there is more than one $T$ axiom box, choose as above. Replace that box with

$$\begin{array}{c}
\hline \\
\rightarrow T \\
\hline
\end{array}$$

Let $t$ be any trip in the proof structure $\beta$ obtained from $\beta_0$ by the above procedure. Then $t$ is the same as some trip $t_0$ in $\beta_0$, except for the extra segment of $t$ around the new terminal $\bot$. It is clear that $t$ must be a long trip if $t_0$ is.

(ii) Suppose $\Pi$ is obtained from $\Pi_0$ by the rule (Th?):

$$
\frac{}{\Gamma \vdash B_1, \ldots, B_n} \quad (\text{Th?})
$$

Assume we have a proof net $\beta_0 = N(\Pi_0)$ with terminals $B_1, \ldots, B_n$. We construct a proof structure $\beta$ from $\beta_0$ by 'inserting a $?A$', following a procedure analogous to the one described in (i). And as above, $\beta$ will be a proof net.

\[\square\]

**Theorem 6.2.7:**

The map $N : LLT \rightarrow PNT$ is surjective.

**Proof (sketch):**

If $\beta$ in $PNT$ consists of an axiom box then choose any one of the equivalent-up-to-exchanges proofs in $LLT$ of the appropriate sequent.

Suppose $\beta$ contains more than one link. We can readily deal with the cases where $\beta$ contains a terminal unary link or a terminal $?B$ or $?C$ link. This leaves us with the hypothesis that all the
terminal links in $\beta$ are either cut links or $\otimes$ links. It is clear that if $A$ is a premise of either a cut link or $\otimes$ link, then (i) the premise of a unary link is in $eA$ iff the conclusion is, and (ii) the conclusion of a $C?$ link is in $eA$ iff both premises are. We have to modify the trip specified in the proof of the Trip Theorem (6.0.20) so as to deal with the case when exactly one premise of a $C?$ link is in $eA$, in addition to the case when exactly one premise of a $\otimes$ link is in $eA$. Provided we always allow for this new possibility, the proof of the Splitting Theorem goes through.

In the same way as was noted in Observation 6.0.10, we can define $\beta^*$, for $\beta$ in PNT, in terms of $\Pi^*$, where $\beta = N(\Pi)$.

6.3 Proof boxes and PNI!

The rules $(\&)$ and $(!)$ of LL! both involve a global constraint which is antithetical to the spirit of localism inherent in proof nets as developed so far. For these rules, the use of proof boxes seems unavoidable.

In Chapter 6 of Girard [1987], some tentative (and inconclusive) suggestions are given regarding the avoidance of $\&$ proof boxes by working instead with families
of proof structures called \textit{slices}.

Another approach might be to have a binary link corresponding to the rule of adjunction used in Hilbert-style formulations of relevance logics (see Avron [1988]). It would be as follows:

if $\beta_0$ and $\beta_1$ are proof nets with $A$ and $B$, respectively, as their sole terminal formulae, then form a proof net as illustrated.

\[
\begin{array}{c}
\beta_0 \\
A \\
\hline \\
\beta_1 \end{array} \begin{array}{c}
B \\
A\&B \\
\end{array}
\]

All other instances of the sequent calculus rule ($\otimes$) would have to be captured using appropriate axiom boxes. Now in a Hilbert-style system, we make use of the axiom scheme

\[(C \to A) \& (C \to B) \to (C \to (A \& B))\]

by deriving an instance of the antecedent using the rule of adjunction and then applying Modus Ponens. To do the same in the context of proof nets would involve essential and ineliminable \textsc{cut}'s against axiom boxes.

\[
\begin{array}{c}
\beta_0 \\
C-A \\
\hline \\
C\&A \end{array} \begin{array}{c}
\beta_1 \\
C-B \\
\hline \\
C\&B \\
\hline \\
(C\&A) \& (C\&B) \end{array} \begin{array}{c}
(C^\otimes A^\otimes) \otimes (C^\otimes B^\otimes) \to C - A\&B \\
\textsc{cut} \end{array}
\]
Finding the alternatives unsatisfactory, we resign ourselves to proof boxes.

**Definition 6.3.0:**

The system of proof nets PNI! is defined inductively as follows.

(i) If $\beta$ is a proof net in PNT then $\beta$ is a proof net in PNI!.

(ii) If $\beta_0$ and $\beta_1$ are proof nets in PNI! with terminal formulae $A, C_1, \ldots, C_n$ and $B, C_1, \ldots, C_n$ respectively then $\beta$ is a proof net in PNI! with terminal formulae $A \& B, C_1, \ldots, C_n$, where $\beta$ is as illustrated:

![Diagram](image)

In this case, $\beta$ is called an $\&$-proof box.

(iii) If $\beta_0$ is a proof net in PNI! with terminal formulae $A, ?B_1, \ldots, ?B_n$ then $\beta$ is a proof net in PNI! with terminal formulae $!A, ?B_1, \ldots, ?B_n$, where $\beta$ is as illustrated:

![Diagram](image)

In this case, $\beta$ is called a $!$-proof box.
The system of proof nets \textbf{PNI} is obtained from \textbf{PNI}! by omitting clause (iii) above, disallowing D? links and C? links, setting $m=0$ in generalized axiom links and 1 axiom boxes (where $m$ is the number of formulae of the form $?A$) and restricting all links to occurrences of formulae of $\mathcal{L}$.

An $\&$-or $!$-proof box in a proof net acts as an impermeable barrier between the proof nets or net inside and the rest of the proof net of which the box is a part. In the region of a proof net outside all of the proof boxes of the net, the concept of a trip still makes sense. From this perspective, the conclusions of proof boxes look and behave as if they were conclusions of axiom boxes, and all other links in the region behave as they do in \textbf{PNT}. Likewise, in the region of a proof net inside a proof box but outside any other proof box, the concept of a trip still makes sense. And an innermost region of a proof net in \textbf{PNI}!, i.e. a region no part of which is outside any proof box, is just a proof net in \textbf{PNT}. So we can have lots of partial trips in a proof net in \textbf{PNI}! but travelling from the inside of a proof box to the outside, or vice-versa, is impossible.

It is possible to give something like a long trip characterization of proof nets in \textbf{PNI}!: partial trips in each of the unconnected regions are long, relative to the region.
Theorem 6.3.1:
If $\Pi$ is a proof of $\vdash A_1, \ldots, A_n$ in $\text{LL}!$
then we can naturally associate with $\Pi$ a proof net $N(\Pi)$ in $\text{PNI}!$ whose terminal formula occurrences are exactly $A_1, \ldots, A_n$.

Proof:
If the last rule used in $\Pi$ is either ($\&$) or ($!$) then the proof net $N(\Pi)$ falls out of Definition 6.3.0.

Theorem 6.3.2:
The map $N : \text{LL}! \rightarrow \text{PNI}!$ is surjective.

Proof (Sketch):
If $\beta$ is either an $\&$-proof box or a $!$-proof box then the proof $\Pi$ such that $\beta = N(\Pi)$ comes directly from the proofs or proof given by the induction hypothesis by either ($\&$) or ($!$).

Again, we are reduced to the case where all the terminal links of $\beta$ are either $\text{CUT}$ links or $\otimes$ links. Now we can work just in the outermost region of $\beta$, considering all proof boxes which border the region as axiom boxes. Then proceed as in the proofs of the corresponding results for $\text{PNO}^+$ and $\text{PNT}$.

As before, we define $\beta^*$ for $\beta$ in $\text{PNI}!$ in terms of $\Pi^*$, where $\beta = N(\Pi)$. 
As we noted earlier, this 'top-down' inductive definition of $\beta^*$ is less than optimal for use in the context of the normalization process. We now present Girard's alternative 'bottom-up' characterization of $\beta^*$, modified to suit our variant treatment of (1) and (1h).

**Definition 6.3.3:**

Let $\beta$ be a proof net in PN1! and fix an ordering, say $C_1, \ldots, C_n$, on the terminal formula occurrences of $\beta$.

A basis for an experiment in $\beta$ is a sequence $W = (W_1, \ldots, W_n)$ such that $W_i \in C_i$ for $i = 1, \ldots, n$.

An experiment in $\beta$ consists of the allocation of one or more elements $x \in |A|$, for each occurrence of a formula $A$ in $\beta$, according to the following procedure.

(i) Arbitrarily select a basis $W \in |C, B, \ldots, C_n|$.

(ii) For each CUT link $\frac{A}{\text{cut}} \rightarrow \frac{A^*}{\text{cut}}$ in $\beta$, arbitrarily select an element $x \in |A| = |A^*|$; we consider this $x$ to be allocated to both $A$ and $A^*$.

(iii) If $\frac{A}{\text{meet}} \rightarrow \frac{B}{\text{meet}}$ is any $\otimes$ link in $\beta$ and we have already allocated $z = (x, y) \in |A \otimes B|$, then allocate $x \in |A|$ and $y \in |B|$.

(iv) If $\frac{A}{\text{meet}} \rightarrow \frac{B}{\text{meet}}$ is any $\otimes$ link in $\beta$ and we have already allocated $z = (x, y) \in |A \otimes B|$, then allocate $x \in |A|$ and $y \in |B|$.

(v) If $\frac{A}{\text{meet}} \rightarrow \frac{B}{\text{meet}}$ is any $\text{fst} \otimes$ link in $\beta$ and we have already allocated $z \in |A \otimes B|$, then if $z = (0, x)$, allocate $x \in |A|$ but if $z = (1, y)$ then abandon the experiment.
(vi) If $\frac{B}{A \otimes B}$ is any $\land \otimes$ link in $\beta$ and we have already allocated $z \in |A \otimes B|$, then if $z = (1,y)$, allocate $y \in |B|$, but if $z = (0,x)$ then abandon the experiment.

(vii) If $\frac{\Delta}{A \& B}$ is any $\land$-proof box in $\beta$ and we have already allocated $z \in |A \& B|$ and $y \in |\Delta|$, then if $z = (0,x)$, allocate $x \in |A|$ and $y \in |\Delta|$, and if $z = (1,y)$, allocate $y \in |B|$ and $y \in |\Delta|$.

(viii) If $\frac{A}{?A}$ is any $\land$ link in $\beta$ and we have already allocated $a \in |?A|$, then if $a = \star x$, allocate $x \in |A|$, but if $a$ is not a singleton then abandon the experiment.

(ix) If $\frac{?A}{?A}$ is any $\land$ link in $\beta$ and we have already chosen $a \in |?A|$, then arbitrarily choose a decomposition $a = a' \cup a''$ and allocate $a' \in |?A|$ for the left premiss and $a'' \in |?A|$ for the right premiss.

(x) Let $\frac{A \& B \cdots \& B_k}{!A \& B \cdots \& B_k}$ be any $!$-proof box in $\beta$ and suppose we have already allocated $a \in |!A|$ and $b_j \in |?B_j|$ for $j = 1, \ldots, k$. Now $a = \{x_1, \ldots, x_m\}$ for some $m > 0$.

For each $j = 1, \ldots, k$, arbitrarily choose a decomposition $b_j = b_{j1} \cup \cdots \cup b_{jm}$. Then for each $i = 1, \ldots, m$, allocate $x_i \in |A|$ and $z_{ji} = (b_{i1}, \ldots, b_{ik}) \in |?B_1, \ldots, ?B_k|$.

In addition to the circumstances already specified, an experiment is to be abandoned if at any stage in the allocation process, we arrive at an occurrence of a formula $A$ such that $|A| = \emptyset$; in particular, if the proof
Net contains either $\top$ or $\bot$.

An experiment in $\beta$ is **completed** when, for each occurrence of a formula $A$ in $\beta$, at least one element $x \in |A|$ has been allocated. If an experiment is abandoned then it cannot be completed.

A completed experiment in $\beta$ is **successful** iff (i) for each generalized axiom link in $\beta$,

$$\begin{array}{c}
\text{L}^k \vdash \exists A_1 \ldots \exists A_m C \vdash C^+ \\
\end{array}$$

if $(\exists, \ldots, \exists, a_1, \ldots, a_m, x, x')$ has been allocated then $a_i = \emptyset$ for $i = 1, \ldots, m$ and $x = x'$; and

(ii) for each $\bot$ axiom box in $\beta$,

$$\begin{array}{c}
\exists A \vdash C \\
\end{array}$$

if $(\exists, a_1, \ldots, a_m)$ has been allocated then $a_i = \emptyset$ for $i = 1, \ldots, m$.

(End of definition 6.3.3)

**Theorem 6.3.4**: (Girard [1987])

Let $\beta$ be a proof net in PNI; and let $C_1, \ldots, C_n$ be an ordered list of the terminal formula occurrences of $\beta$.

Write $C_1, \ldots, C_n$ as $\Gamma$. Then

$$\beta^* = \{ \forall \in |\Gamma| \mid \forall \text{ is a basis for a successful experiment in } \beta \}$$

**Proof (sketch):**

Suppose $\forall \in |\Gamma|$ is a basis for a successful experiment in $\beta$. This means that whenever $\beta_0$ is a generalized axiom link with conclusions $L^k \vdash \exists x, C, C^+$ and $z \in |L^k \vdash \exists x, C, C^+|$ is one of the sequences allocated on the basis of $\forall$, we have $z \in \beta^*$. And likewise whenever $\beta_0$ is a $\bot$ axiom box. Since $\forall$ is a basis for a successful experiment in
β, there can be no T axiom boxes in β. So W determines correct sequences for all the axiom boxes in β. By following the course of these sequences through the 'top-down' inductive definition of β* (via the appropriate Π*
, where β = N(Π)) we see that W ∈ β*.

Conversely, if W ∈ β* then it is immediate that W is a basis for a successful experiment in β.

Note that if W ∈ β* then β* ≠ ∅, hence β cannot contain any T axiom boxes. When β does contain a T axiom box, there are no successful experiments so the set of all bases for successful experiments in β is empty.

To ensure that we have identical subgraphs of the web W(Π) and not just identical sets, we have to establish that

\{ W ∈ |Π| | W is a basis for a successful experiment in β \}

is a coherent subset of |Π|. The result is proved for proof nets β in PNO in Girard [1987] (Theorem 3.18). The difficulties are concentrated in the multiplicative subsystem; in comparison, the extension to PNI! is fairly straightforward.
6.4 Normalization in $\text{PN1}$!

In what follows, we shall define a number (twelve, to be precise) of conversion schemes. We say $\beta$ 
converts to $\beta'$, written $\beta \cong \beta'$, iff the transition from $\beta$ to $\beta'$ is an instance of one of these schemes. The definitions of the other terms appropriate to the discussion of normalization carry over from Section 6.1.

For all but one of the conversion schemes, the size of the resulting proof net is strictly smaller than the size of the original. The troublesome case is when one premise $!A$ of a cut link is a conclusion of a $!$ proof box and the other premise $?A^+$ is the conclusion of a $?!$ link; the required conversion produces an exponential growth in size.

**Definition 6.4.0:**
We say a proof net $\beta$ in $\text{PN1}$! is built from boxes $\Box_1, \ldots, \Box_n$ and formula occurrences $A_1, \ldots, A_k$ by means of the unary and binary links iff (1) each $\Box_i$, $i=1, \ldots, n$, is either an axiom box or a proof box which is not inside any other box, and $\Box_1, \ldots, \Box_n$ is an exhaustive list of all such boxes, and (2) each $A_j$, $j=1, \ldots, k$, is the conclusion of either a unary or a binary link and is not inside any proof box, and $A_1, \ldots, A_k$ is an exhaustive list of such formula occurrences. (So the conclusions of an outermost box are not counted.)

The size $s(\beta)$ of a proof net $\beta$ is defined by induction on the build-up of $\beta$ as follows:
(i) If $\beta$ is built from boxes $\square_1, \ldots, \square_n$ and formula occurrences $A_1, \ldots, A_k$ by means of the unary and binary links, then

$$s(\beta) = s(\square_1) \cdot \ldots \cdot s(\square_n) \cdot 2^k.$$  

(ii) If $\beta$ is a (proof or axiom) box then the size of $\beta$ is defined as follows:

- If $\beta$ is a generalized axiom link with $k+m+2$ conclusions (as in Definition 6.2.4), then $s(\beta) = k+m+3$;
- If $\beta$ is a $\bot$ axiom box with $m+1$ conclusions (as in Definition 6.2.4), then $s(\beta) = m+2$;
- If $\beta$ is a $T$ axiom box with $m+1$ conclusions (as in Definition 6.2.4) then $s(\beta) = m+2$;
- If $\beta$ is an $\&$-proof box formed from proof nets $\beta_0$ and $\beta_1$, then $s(\beta) = s(\beta_0) + s(\beta_1) + 1$;
- If $\beta$ is a $!$-proof box formed from a proof net $\beta_0$, then $s(\beta) = s(\beta_0) + 1$.

To distinguish between the different sorts of conclusions of an axiom or proof box, we say the front door(s) of a generalized axiom link, $\bot$ axiom box and $T$ axiom box are the atomic formulae $C$ and $C^+$, $\bot$ and $T$ respectively; all the other conclusions of such boxes are called side doors. Likewise, the front door of an $\&$-proof box ($!$-proof box) is $A \& B$ ($!A$) and all the other conclusions are called side doors of the box.
The conversion schemes are of three kinds.

1. **Generalized axiom conversion.** This is performed any time we have a CUT link, both premises of which are front doors of generalized axiom links. This is the analogue of the scheme (AC) defined in 6.1.0.

2. **Symmetric conversions.** These cover cases where the premises of a CUT link come by dual links.

3. **Commutative conversions.** These are required when one of the premises of a CUT link is a side door of either a T axiom box, an &-box or a !-box. (Side doors of generalized axiom links and 1 axiom boxes are dealt with in (2).)

The conversions required for the modal connectives are such that reduction in PNI! fails to have the Church-Rosser property. The property of preserving semantic objects, that is

\[ \text{if } \beta \sim \beta_0 \text{ then } \beta^* = \beta_0^* \]

is the (not to be scoffed at) consolation prize; a proof net \( \beta \) in PNI! may reduce to distinct normal forms \( \beta_0^* \) and \( \beta_1^* \), but since \( \beta_0^* = \beta_1^* \), they do not differ too much.

We will use the 'bottom-up' characterization of \( \beta^* \) to verify, for some but not all of the conversion schemes, that if \( \beta \text{ con } \beta_0 \) then \( \beta^* = \beta_0^* \). For the cases omitted, consult the original: Girard [1987], Chapter 4.
(1) **Generalized axiom conversion (GAC):**

\[ \frac{1^k - \Delta - C - C^+}{C - C^+ - ?\Gamma - 1^j} \]

is obtained from \( \beta \) by replacing a configuration of the form

\[ \frac{1^k - 1^j - ?\Delta - ?\Gamma - C - C^+}{\text{CUT}} \]

by

\[ \frac{1^k - 1^j - ?\Delta - ?\Gamma - C - C^+}{\text{CUT}} \]

if \( s(\beta) = x(k + m + 3)(j + n + 3) \)

then \( s(\beta') = x(k + j + m + n + 3) \).

We claim \( \beta^* = \beta'^* \).

This is proved by induction on the depth of nesting of boxes containing the CUT link in question.

Base case: \( n = 0 \). So the CUT link is in the outermost region of \( \beta \), i.e., it is not contained in any box. Let \( \tilde{\mathcal{W}} \) be any basis for a successful experiment in \( \beta \).

Suppose we have chosen \( x \in 1c1 \) and \( x \in 1c^+1 \) for the CUT link in \( \beta \). Since the experiment succeeds on the basis of \( \tilde{\mathcal{W}} \), \( \tilde{\mathcal{W}} \) must force the allocation of \( x \in 1c1 \) for \( C \) in the left axiom box and \( x \in 1c^+1 \) for \( C^+ \) in the right axiom box, as well as ensuring that \( \phi \) is assigned to each \( ?A_i \) in \( ?\Delta \) and to each \( ?B_j \) in \( ?\Gamma \). (Note that \( @ \) is always allocated to \( 1 \) since \( |1| = \{ @ \}. \))

But in such a case, \( \tilde{\mathcal{W}} \) is also the basis of a successful experiment in \( \beta' \). Hence \( \beta^* \in \beta'^* \). Conversely, suppose \( z \) is a basis for a successful experiment in \( \beta' \).
Then $z$ must force the allocation of $x \in |C|$ and $x \in |C+1|$ as well as ensuring that all the $\theta_1$ and $\theta_2$ are assigned $\phi$. Now with regard to the CUT link in $\beta$, we are free to choose $x \in |C|$ and $x \in |C+1|$ for the premises $C$ and $C+1$. So $z$ is a basis for a successful experiment in $\beta$. Hence $\beta^* \subseteq \beta^*$.

Induction step: $n > 0$. So the conversion takes place within a box $\square$, and $\square$ is replaced by $\square'$. If $\square$ is not the innermost proof box containing the CUT then consider the box $\square_0$ of depth one fewer than the depth of $\square$. The conversion replaces $\square_0$ with $\square_0'$, and the induction hypothesis gives us $\square_0^* = (\square_0')^*$. Since the conversion takes place inside $\square_0$, it has no impact outside that box; in particular, the constituents of $\square$ other than $\square_0$ are the same as the constituents of $\square'$ other than $\square_0'$. Hence $\square^* = \square'^*$. If $\square$ is the innermost proof box containing the CUT then there is a proof net $\beta_0$ inside $\square$ such that the conversion replaces $\beta_0$ with $\beta_0'$. By the base case of the induction, $\beta_0^* = (\beta_0')^*$, and so $\square^* = \square'^*$. Now $\beta'$ is built from $\square'$ in the same way that $\beta$ is built from $\square$, so we must have $\beta^* = \beta'^*$.

The argument for the induction step given above applies generally to any conversion. So for the remaining conversions, we need only consider the base case of the induction, i.e. we may assume the conversion does not take place within a box.
The conversion scheme (GAC) deals with the case when both premises of a cut link are front doors of a generalized axiom link. If exactly one premise of a cut link is such a front door then, since it is an atomic formula, the only way in general that the dual atomic formula can occur is as a side door of a T axiom box. That class of cuts is dealt with in (3.1) below.

(2) Symmetric conversions

(2.1) Multiplicative symmetric conversion \((\otimes/\&-\text{sc})\):

\[ p' \] is obtained from \( p \) by replacing a configuration of the form

\[
\begin{array}{c}
\otimes \\
A \otimes B \\
A^t \otimes B^t \\
\text{cut}
\end{array}
\]

by

\[
\begin{array}{c}
\otimes \\
A \otimes B \\
A^t \otimes B^t \\
\text{cut}
\end{array}
\]

If \( s(p) = x \cdot 2^6 \) then \( s(p') = x \cdot 2^4 \).

In any experiment in \( p \), we have to select a pair \((x, y) \in |A \otimes B| = |A^t \otimes B^t|\) then allocate \( x \in |A| \) and \( x \in |A^t| \), and \( y \in |B| \) and \( y \in |B^t| \). But this is indistinguishable from an experiment in \( p' \), where we select \( x \in |A| = |A^t| \) and \( y \in |B| = |B^t| \). Hence \( \omega \) is a basis for a successful experiment in \( p \) iff it is a basis for a successful experiment in \( p' \). Hence \( p^* = p'^* \).
(2.2) **Multiplicative Unit Conversion** \((1/\bot - \text{sc})\)

\(\beta'\) is obtained from \(\beta\) by replacing a configuration of the form

\[
\begin{array}{c}
\text{cut} \\
\hline
\text{cut} \\
\hline
?\Gamma - 1_k \\
\hline
1_k - ?\Delta - ?C - ?C^+ \\
\hline
\end{array}
\]

by

\[
\begin{array}{c}
\text{cut} \\
\hline
\text{cut} \\
\hline
1_k - ?\Gamma - ?\Delta - ?C - ?C^+ \\
\hline
\end{array}
\]

If \(s(\beta) = x(n+2)((k+1)+m+3)\) then \(s(\beta') = x(k+n+m+3)\)

In \(\beta\), we have to select \(\varepsilon \in \{1\} = \{\bot\}\) but this contributes nothing to the success or otherwise of an experiment. What matters is the allocation of elements \(a \in \{A\} \in \{\Gamma\}, \ b_j \in \{B\} \in \{\Delta\}, \ x \in \{C\}\) and \(x' \in \{C^+\}\), just as in an experiment in \(\beta'\). Hence \(\beta^* = \beta'^*\)

(2.3) **Additive Symmetric Conversion** \((\delta / \text{fst@ - sc})\)

\(\beta'\) is obtained from \(\beta\) by replacing a configuration of the form

\[
\begin{array}{c}
\text{cut} \\
\hline
\hline
\hline
\end{array}
\]

If \(s(\beta) = (x+y+1) \cdot 2z\) then \(s(\beta') = x \cdot z\)
Suppose that in an experiment in β, we have \( z \in |\Delta| \) and we have to choose an element in \(|A \land B| = |A^* \oplus B^*|\). If we choose \((1, y) \in |A \land B|\) then, due to the \( \text{FST} \oplus \) link, the experiment will not succeed. So in successful experiments in \( \beta \), an element \((0, x) \in |A \land B|\) is chosen and \( z, x \) is allocated to \( \Delta, A \) and \( x \) is allocated to \( A^* \). In an experiment in \( \beta' \) where \( z \in |\Delta| \) is given, we have to choose an \( x \in |A| = |A^*| \) then allocate \( z, x \) to \( \Delta, A \) and \( x \) to \( A^* \). Hence \( \beta^* = \beta'^* \).

\[(2.4) \quad \text{Additive symmetric conversion} \ (\& / \text{SN}D \oplus - \text{SC}):\]

the obvious modification of \((\& / \text{FST} \oplus - \text{SC})\).

\[(2.5) \quad \text{Additive unit conversion} \ (\& / T - \text{SC}):\]

Although there is no link specifically for \( \& \), we allow constants as well as literals to be front doors of generalized axiom links.

\( \beta' \) is obtained from \( \beta \) by replacing a configuration of the form

\[
\begin{array}{c}
\frac{\neg \Delta \rightarrow T \rightarrow O}{\text{CUT}} \\
\end{array}
\]

by

\[
\begin{array}{c}
\neg T \rightarrow \neg \Delta \rightarrow \neg \Sigma \rightarrow O \\
\end{array}
\]

If \( s(\beta) = x(k + m + 3)(n + 2) \) then \( s(\beta') = x(k + m + n + 2) \).

\( \beta^* = \beta'^* = \emptyset \).
(2.6) Exponential symmetric conversion ($! I / n ? - S C$):
This conversion is required whenever one premise of a cut link is a side door $?A$ of either a generalized axiom link or a $1$ axiom box. We state it for the latter.

$p'$ is obtained from $p$ by replacing a configuration of the form

```
  ![Diagram]
```

by

```
  ![Diagram]
```

If $s(p) = \pi \cdot (y+1)(m+3)$ then $s(p') = \pi \cdot (n+m+2)$
for some $n \leq y$.

In an experiment in $p$, suppose we have $b \in 1 ?A$ and
we need to select an $a \in 1 !A = 1 ?A^4$. For the experiment
to succeed, we must choose $a = \phi$. But if $a = \phi$ then we
must have $b_i = \phi$, $i = 1, \ldots, n$, where $b = (b_1, \ldots, b_n)$. This
is what is required for a successful experiment in $p'$.

(2.7) Exponential symmetric conversion ($! D / ? - S C$

$p'$ is obtained from $p$ by replacing a configuration of the form

```
  ![Diagram]
```

by

```
  ![Diagram]
```
If \( s(\beta) = (x+1) \cdot 4z \) then \( s(\beta') = x \cdot 2z \).

In an experiment in \( \beta \), suppose we have \( k \in |\Delta| \) and we need to select an \( a \in |A| \cdot |A^*| = |A^*| \cdot 2^{|A^*|} \). Due to the \( \Delta \) link, the experiment will be abandoned unless \( a = \alpha \). Then we assign \( \alpha \) to \( A^* \) and \( k, \alpha \) to \( \Delta, A \). (Since \( a \) is a singleton, we don't have to choose a decomposition of \( k \).) But this is what we do in any experiment in \( \beta' \). Hence \( \beta^* = \beta'^* \).

\[
\text{(2.8) Exponential Symmetric conversion (I/c? - sc)}:
\]
\( \beta' \) is obtained from \( \beta \) by replacing a configuration of the form

![Diagram](image)

The duplication of the \( ! \)-proof box results in an exponential growth in size.

If \( s(\beta) = (x+1) \cdot 8z \)
then \( s(\beta') = (x+1)^2 \cdot 4z \cdot 2^n \) where \( n < x \) (\( n \) is the number of formula occurrences in \( \Delta \)).

\[
\text{(3) Commutative conversions.}
\]
To deal with the cases when one of the premises of a \( \text{cut} \) link is a side door of a \( T \)-axiom box or an \&-box, we need the concept of a ghost box.
Let $\frac{B}{\text{cut}} \circ B^*$ be a cut link, in a given proof net $\beta$, which is not inside any proof box. In this outermost region of $\beta$, we can treat the conclusions of any proof box bordering the region as if they were the conclusions of an axiom box. So we can sensibly consider the empire $eB$ of $B$. The frontier of $eB$ consists of $B$ together with the conclusions of $\beta$ which belong to $eB$ and the premises of $B$ links and $C$? links such that the other premise is not in $eB$. Otherwise put, the frontier of $eB$ consists of those formula occurrences in $eB$ that become terminal when we consider $eB$ as (the underlying set of) a proof net. List the frontier of $eB$ as $\Delta, B$. The ghost box of $eB$ is an object of the form

$$\boxed{eB \quad \Delta \quad B \quad J}$$

For the purposes of tram trips (within this outermost region of $\beta$), $\Delta, B$ behave like the conclusions of an axiom box. The materialization of $eB$ consists of replacing $eB$ in $\beta$ by the ghost box of $eB$.

Proposition 6.4.1: (Girard [1987])

If $\beta$ is a proof net in $\text{PN1}$! and $\frac{B}{\text{cut}} \circ B^*$ is a cut link in $\beta$ not inside any proof box, then the proof structure obtained from the outermost region of $\beta$ by materializing $eB$ and $eB^*$ does not admit any short-trips.
For the purposes of conversion, ghost boxes give us much needed information about a premise of a \textit{Cut} link. The formula occurrences in the frontier of \( eB \), other than \( B \) itself, are those which are both connected with \( B \), in the sense that they lie in the interval \([B^\alpha, B^\nu]_t\) for each \( t \), and are linked with the region outside \( eB \).

\[(3.1) \text{ Zero commutation (T-cc)} \]

If \( \beta \) contains the configuration

\[
\begin{array}{c}
\text{T} - \Gamma - A \\
\text{cut}
\end{array}
\]

then first materialize \( eA^\Lambda \),

\[
\begin{array}{c}
\text{T} - \Gamma - A \\
\text{cut}
\end{array}
\]

then create \( \beta' \) by replacing the above configuration by one of the form

\[
\begin{array}{c}
\text{T} - \Gamma - A \\
\text{cut}
\end{array}
\]

If \( s(\beta) = (n+3) \cdot x \cdot y \) then \( s(\beta') = (n+m+2) \cdot y \)

where \( m < x \).

\( \beta^* = \beta'^* = \emptyset \).
(3.2) **Additive commutation** $(\& - \text{cc})$:

If $\beta$ contains the configuration

```
<table>
<thead>
<tr>
<th>\beta_0</th>
<th>\beta_1</th>
</tr>
</thead>
<tbody>
<tr>
<td>A-\Gamma-C</td>
<td>B-\Gamma-C</td>
</tr>
</tbody>
</table>

\[ \text{A&B-}\Gamma-C \quad \text{cut} \]
```

then first materialize $eC^\perp$,

```
<table>
<thead>
<tr>
<th>\beta_0</th>
<th>\beta_1</th>
</tr>
</thead>
<tbody>
<tr>
<td>A-\Gamma-C</td>
<td>B-\Gamma-C</td>
</tr>
</tbody>
</table>

\[ \text{A&B-}\Gamma-C \quad \text{cut} \]
```

then create $\beta'$ by replacing the above configuration by one of the form

```
<table>
<thead>
<tr>
<th>\beta_0</th>
<th>\beta_1</th>
</tr>
</thead>
<tbody>
<tr>
<td>A-\Gamma-C</td>
<td>B-\Gamma-C</td>
</tr>
</tbody>
</table>

\[ \text{A&B-}\Gamma-C \quad \text{cut} \]
```

After performing this conversion, the ghost boxes may be erased.

If $s(\beta) = (x+y+1) \cdot z \cdot t$ then $s(\beta') = (xz + yz + 1) \cdot t$
(3.3) Exponential commutation (¬-CC)

In the case when one premise of a CUT link is a side door of a !-proof box, ghost boxes are of no help. If ?C is one premise of a CUT then the other is !C. If !C is a side door of a T axiom box then use the scheme (T-CC), and if !C is a side door of an &-proof box then use the scheme (&-CC). The only remaining possibility is that !C is the front door of a !-proof box.

\( \beta' \) is obtained from \( \beta \) by replacing a configuration of the form

\[
\begin{array}{c}
\beta_0 \\
?\Delta - C \\
\hline \\
?\Delta - !C \\
\hline \\
\end{array}
\]

by

\[
\begin{array}{c}
\beta_1 \\
?C - ?\Gamma - A \\
\hline \\
?C - ?\Gamma - !A \\
\hline \\
\end{array}
\]

If \( S(\beta) = (x+1) \cdot (y+1) \cdot Z \) then \( S(\beta') = ((x \cdot (y+1)) + 1) \cdot Z \).
Theorem 6.4.2: The Small Normalization Theorem (Girard [1987])

If \( \beta \) is any proof net in \( \text{PN1}! \) which does not contain a configuration of the form

\[
\begin{array}{c}
\vdots \\
\hline
\vdots \\
\hline
\vdots \\
\hline
\end{array}
\]

\[
\begin{array}{cc}
?A \rightarrow A & \vdots \\
?A & ?A \rightarrow A \\
\vdots & \text{cut}
\end{array}
\]

then \( \beta \) is strongly normalizable.

Proof:

If \( \beta \) does not contain the objectionable configuration then \( L(\beta) < s(\beta) \).

For the proof of the Strong Normalization theorem for \( \text{PN2} \) (and \( \text{PN1}! \)), Girard adapts his technique of 'candidats de réductibilité' originally devised for the system \( \text{F} \). The CR's are sets of proof nets which have an underlying structure similar to that of facts in the phase space semantics. Those readers still breathing are encouraged to first take a long walk and then, suitably refreshed, work through the remaining material on their own.
### Appendix A: A guide to notation

<table>
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<td>$A \rightarrow B$</td>
<td>$A \rightarrow B$</td>
<td>$A \rightarrow B$</td>
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<tr>
<td><strong>negation</strong></td>
<td>$A^\perp$</td>
<td>$\neg A$</td>
<td>$\neg A$</td>
<td>$\neg A$</td>
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<td>$A \otimes B$</td>
<td>$A \otimes B$</td>
<td>$A \otimes B$</td>
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<td></td>
<td>$1$</td>
<td>$I$</td>
<td>$t$</td>
<td>$t$</td>
</tr>
<tr>
<td><strong>par/fission</strong></td>
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<td>$A \uplus B$</td>
<td>$A \uplus B$</td>
<td>$A \uplus B$ or $A \lor B$</td>
</tr>
<tr>
<td></td>
<td>$I$</td>
<td>$\varnothing$</td>
<td>$f$</td>
<td>$f$</td>
</tr>
<tr>
<td><strong>With/and</strong></td>
<td>$A &amp; B$</td>
<td>$A &amp; B$</td>
<td>$A &amp; B$</td>
<td>$A &amp; B$ or $A &amp; B$</td>
</tr>
<tr>
<td></td>
<td>$T$</td>
<td>$T$</td>
<td>$T$</td>
<td>$T$</td>
</tr>
<tr>
<td><strong>plus/or</strong></td>
<td>$A \oplus B$</td>
<td>$A + B$</td>
<td>$A \lor B$</td>
<td>$A \lor B$</td>
</tr>
<tr>
<td></td>
<td>$O$</td>
<td>$O$</td>
<td>$F$</td>
<td>$F$</td>
</tr>
<tr>
<td><strong>&quot;of course&quot;/necessity</strong></td>
<td>$!A$</td>
<td>$!A$</td>
<td>$\square A$</td>
<td>$\square A$</td>
</tr>
<tr>
<td><strong>&quot;why not&quot;/possibility</strong></td>
<td>$?A$</td>
<td>$?A$</td>
<td>$\Diamond A$</td>
<td>$\Diamond A$</td>
</tr>
</tbody>
</table>
Appendix B: Two sided sequent calculi

Let \( \mathcal{L}_0 \) be the collection of formulae generated from propositional letters and constants \( \top, \bot, \top_0, \top_0 \) by means of the unary connective \( (-)^+ \), the binary connectives \( \otimes, \& \) and \( \# \), and the unary connectives \( ! \) and \( ? \).

The two-sided sequent calculus for propositional linear logic, \( \mathbf{DLL} \), consists of the axiom schemes:

\[
\text{(identity)} \quad \Gamma \vdash A, A
\]

\[
\text{(R1)} \quad \top \vdash \top
\]

\[
\text{(RT)} \quad \Gamma \vdash \Gamma, \Delta
\]

\[
\text{(Rneg)} \quad A \vdash A^+\!
\]

\[
\text{(L1)} \quad \bot \vdash \bot
\]

\[
\text{(L0)} \quad \Gamma, \top_0 \vdash \Delta
\]

\[
\text{(Lneg)} \quad A^+\! \vdash A
\]

and the following rules of inference:

**Structural rules:**

\[
\frac{\Gamma \vdash \Delta}{\tau(\Gamma) \vdash \sigma(\Delta)} \quad \text{(Exch)}
\]

where \( \tau \) and \( \sigma \) are any permutations

**Negation rule:**

\[
\frac{\Gamma, A \vdash B, \Delta}{\Gamma, B^+\! \vdash A^+, \Delta} \quad \text{(Var)}
\]

**Additive rules:**

\[
\frac{\Gamma, A \vdash \Delta}{\Gamma, A \& B \vdash \Delta} \quad \text{(L1)}
\]

\[
\frac{\Gamma, B \vdash \Delta}{\Gamma, A \& B \vdash \Delta} \quad \text{(L2)}
\]

\[
\frac{\Gamma \vdash A, \Delta \quad \Gamma \vdash B, \Delta}{\Gamma \vdash A \& B, \Delta} \quad \text{(R1)}
\]

\[
\frac{\Gamma, A \vdash \Delta \quad \Gamma, B \vdash \Delta}{\Gamma, A \oplus B \vdash \Delta} \quad \text{(L\oplus)}
\]

\[
\frac{\Gamma \vdash A, \Delta}{\Gamma \vdash A \oplus B, \Delta} \quad \text{(R\oplus1)}
\]

\[
\frac{\Gamma \vdash B, \Delta}{\Gamma \vdash A \oplus B, \Delta} \quad \text{(R\oplus2)}
\]
multiplicative rules:
\[ \Gamma, \bot \vdash \Delta \quad (\bot) \]
\[ \Gamma \vdash \Delta \quad (R \bot) \]
\[ \Gamma, A \otimes B \vdash \Delta \quad (L \otimes) \]
\[ \Gamma, A, B \vdash \Delta \quad (R \otimes) \]
\[ \Gamma, \neg A \vdash \Delta \quad (L \neg) \]
\[ \Gamma, A \otimes B \vdash \Delta \quad (R \neg) \]
\[ \Gamma, A \otimes B \vdash \Delta \quad (L \otimes) \]
\[ \Gamma, B \vdash \Delta \quad (R \otimes) \]
\[ \Gamma, A \rightarrow B \vdash \Delta \quad (L \rightarrow) \]
\[ \Gamma, A \rightarrow B \vdash \Delta \quad (R \rightarrow) \]

The language for intutionistic linear logic is as above, except that the constant \( \bot \) and the binary connective \( \otimes \) are omitted. The sequent calculus for propositional intutionistic linear logic is obtained from DLL by (i) omitting the axiom schemes \((L \bot), (R \neg)\) and \((L \neg)\); (ii) omitting the inference rules \((\text{VAR}), (R \bot), (L \otimes)\) and \((R \otimes)\); and (iii) stipulating that sequents contain at most one formula on the RHS.

The sequent calculus DLL! is obtained from DLL by adjoining the following inference rules:

\[ \Gamma \vdash \Delta \quad (L \text{Th}) \]
\[ \Gamma \vdash \Delta \quad (R \text{Th}) \]
\[ \Gamma, ! A \vdash \Delta \quad (L \text{C!}) \]
\[ \Gamma, ? A, ? A \vdash \Delta \quad (R \text{C?}) \]
\[
\frac{\Gamma, A \vdash \Delta}{\Gamma, !A \vdash \Delta} \quad \text{(LD!)} \quad \frac{\Gamma \vdash A, \Delta}{\Gamma \vdash ?A, \Delta} \quad \text{(RD?)}
\]

\[
\frac{!\Gamma, A \vdash ?\Delta}{!\Gamma, ?A \vdash ?\Delta} \quad \text{(L?)} \quad \frac{!\Gamma \vdash A, ?\Delta}{!\Gamma \vdash !A, ?\Delta} \quad \text{(R!)}
\]
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