

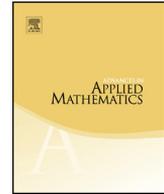


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On 1324-avoiding permutations



Andrew R. Conway, Anthony J. Guttman^{*}

*ARC Centre of Excellence for Mathematics and Statistics of Complex Systems,
Department of Mathematics and Statistics, The University of Melbourne,
Victoria 3010, Australia*

ARTICLE INFO

Article history:

Received 19 August 2014

Accepted 29 November 2014

Available online 23 December 2014

MSC:

05A05

05A15

05A16

Keywords:

Permutation patterns

Enumeration algorithm

Asymptotics

ABSTRACT

We give an improved algorithm for counting the number of 1324-avoiding permutations, resulting in 5 further terms of the generating function. We analyse the known coefficients and find compelling evidence that unlike other classical length-4 pattern-avoiding permutations, the generating function in this case does not have an algebraic singularity. Rather, the number of 1324-avoiding permutations of length n behaves as

$$B \cdot \mu^n \cdot \mu_1^{n^\sigma} \cdot n^g.$$

We estimate $\mu = 11.60 \pm 0.01$, $\sigma = 1/2$, $\mu_1 = 0.040 \pm 0.0015$, $g = -1.1 \pm 0.2$ and $B = 7 \pm 1.3$.

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1. Introduction

Let π be a permutation on $[n]$ and τ be a permutation on $[k]$. Then τ is said to occur as a *pattern* in π if for some subsequence of π of length k all the elements of the subsequence occur in the same relative order as do the elements of τ . For example, 1324

^{*} Corresponding author.

E-mail addresses: andrew1324@greatcactus.org (A.R. Conway), guttman@unimelb.edu.au (A.J. Guttman).

occurs as a pattern in 152364 as 1526 and 1536 as both are in the same relative order as 1324. If a permutation τ does not occur in π , then this is said to be a *pattern-avoiding permutation*, or PAP.

Let $P(z) = \sum_{n \geq 0} p_n z^n$ be the ordinary generating function (OGF) for the number of permutations p_n of length n avoiding the pattern 1324. It is well known that, for the classical, length 4 PAPs, the 24 possible patterns fall into one of three classes [26], called Wilf classes. That is to say, there are three distinct OGFs describing all 24 patterns.

For the sequence 1234 and its associated patterns, in 1990 Gessel [14] showed that the number of length $n > 0$ pattern-avoiding permutations is

$$p_n(1234) = \frac{1}{(n+1)^2(n+2)} \sum_{k=0}^n \binom{2k}{k} \binom{n+1}{k+1} \binom{n+2}{k+1}. \tag{1}$$

Asymptotically,

$$p_n(1234) \sim \frac{81\sqrt{3}}{16\pi} \cdot 9^n \cdot n^{-4},$$

and the generating function $P_{1234}(x) = \sum_n p_n(1234)x^n$ satisfies the linear ODE

$$\begin{aligned} (9x^5 - 19x^4 + 11x^3 - x^2) \cdot \frac{d^3 P_{1234}(x)}{dx^3} + (72x^4 - 153x^3 + 90x^2 - 9x) \cdot \frac{d^2 P_{1234}(x)}{dx^2} \\ + (126x^3 - 264x^2 + 154x - 16) \cdot \frac{dP_{1234}(x)}{dx} \\ + (32 - 72x + 36x^2) \cdot P_{1234}(x) = 0, \end{aligned} \tag{2}$$

with initial conditions $P_{1234}(0) = 1, P'_{1234}(2) = 0, P''_{1234}(2) = 12$.

For the sequence 1342 and its associated patterns, in 1997 Bóna [3] showed that the number of length $n > 0$ pattern-avoiding permutations is

$$\begin{aligned} p_n(1342) = (-1)^{n-1} \cdot \frac{(7n^2 - 3n - 2)}{2} \\ + 3 \sum_{k=0}^n (-1)^{n-i} \cdot 2^{i+1} \cdot \frac{(2i-4)!}{i!(i-2)!} \cdot \binom{n-i+2}{2}. \end{aligned} \tag{3}$$

The generating function $P_{1342}(x) = \sum_n p_n(1342)x^n$ satisfies the linear ODE

$$\begin{aligned} (8x^2 + 7x - 1) \cdot \frac{d^2 P_{1342}(x)}{dx^2} + (28x - 8) \cdot \frac{dP_{1342}(x)}{dx} + 12 \cdot P_{1342}(x) = 0, \\ P_{1342}(0) = 1, \quad P'_{1342}(0) = 1. \end{aligned} \tag{4}$$

Indeed, it can be exactly solved to give [22] the simple algebraic expression

$$P_{1342}(x) = \frac{32x}{1 + 20x - 8x^2 - (1 - 8x)^{3/2}} = \frac{(1 - 8x)^{3/2}}{2(1 + x)^3} + \frac{(1 + 20x - 8x^2)}{2(1 + x)^3},$$

from which one readily obtains

$$p_n(1342) \sim \frac{64}{243 \cdot \sqrt{\pi}} \cdot 8^n \cdot n^{-5/2}.$$

It has been pointed out by J.-M. Maillard (private communication) that the third-order differential operator (2) is in fact a direct sum of two lower order operators, $L1 + M2$, where $L1$ is a first-order operator with a rational solution, and $M2$ is a second-order operator which is Fuchsian, globally nil-potent and corresponds to a modular form. The second-order differential operator (4) is the direct sum of two first-order operators, whose solutions are given by the two terms on the right hand side of the equation for the solution, given above.

The remaining class, that of 1324-avoiding permutations, remains unsolved [27]. Even the growth constant is not accurately known. The best upper bound is 13.73718 [4], due to Bóna, while Claesson, Jelínek and Steingrímsson [7] gave an improved bound, $e^{\pi\sqrt{2/3}} \approx 13.00195$, but subject to the validity of an unproved conjecture. The best published lower bound was 9.47, proved by Albert et al. [1] but Bevan [2] has very recently improved this to 9.81. Careful Monte Carlo work by Madras and Liu [23] implies that the growth constant lies in the range [10.71, 11.83].

In this paper we give details of an improved algorithm for the enumeration of such PAPs, with which we obtained five further terms in the OGF beyond the existing longest known sequence, due to Johansson and Nakamura [21], using comparable computing resources. We will refer to their algorithm as the JN algorithm.

We also analyse the sequence of coefficients, and provide compelling numerical evidence that the asymptotic form of the coefficients is more complex than that of the two solved classes just considered. The generating functions of the two solved cases have algebraic singularities, whose coefficients have the asymptotic form $p_n \sim D \cdot \mu^n \cdot n^g$. Our numerical studies, detailed below, lead us to suggest that the coefficients $p_n(1324) \sim B \cdot \mu^n \cdot \mu_1^{n^\sigma} \cdot n^g$, where $\sigma = \frac{1}{2}$.

Many enumeration problems in algebraic combinatorics have generating functions with algebraic singularities, and hence coefficients with leading asymptotic form

$$a_n \sim A \cdot \mu^n \cdot n^g, \tag{5}$$

where $1/\mu$ is the radius of convergence, sometimes called the *critical point*, g is a critical exponent and A is a critical amplitude. However, a number of solved, and, we believe, unsolved problems that arise in both algebraic combinatorics and mathematical physics have a more complex singularity structure, with a sub-dominant asymptotic term $O(\mu_1^{n^\sigma})$ rather than $O(n^g)$. In fact the sub-sub-dominant term is of $O(n^g)$. That is to say, the dominant asymptotic behaviour of the coefficients b_n of the associated generating function is

$$b_n \sim B \cdot \mu^n \cdot \mu_1^{n^\sigma} \cdot n^g. \tag{6}$$

Perhaps the best-known example of this sort of behaviour is the number of partitions of the integers – though in that case the leading exponential growth term μ^n is absent (or, equivalently, $\mu = 1$). Another example is the generating function for the number of fragmented permutations [12], which is

$$F(z) = \exp\left(\frac{z}{1-z}\right).$$

Then, with $F_n = [z^n]F(z)$, we have [12, p. 563]

$$F_n \sim \frac{e^{2\sqrt{n}}}{2\sqrt{\pi en^{3/4}}}.$$

These two examples highlight the fact that singularities of quite different analytic structure can give rise to the same asymptotics. The OGF for integer partitions has a natural boundary at its radius of convergence, whereas the generating function for fragmented permutations is D-finite. So if such an asymptotic form (6) is observed, one cannot say much about the underlying singularity structure from the asymptotics alone.

Another example, discussed in [16], is the problem of Dyck paths counted not only by length but also by height h , which is defined to be the maximal vertical displacement of the Dyck path from its horizontal axis. Let $d_{n,h}$ be the number of Dyck paths of length $2n$ and height h . The generating function is

$$D(x, y) = \sum_{n,h} d_{n,h} x^{2n} y^h.$$

Then

$$[x^{2n}]D(x, y) = \sum_{h=1}^n d_{n,h} y^h \sim B \cdot 4^n \cdot \mu_1^{n^{1/3}} \cdot n^{-5/6}, \tag{7}$$

where both $B(y)$ and $\mu_1(y)$ are known [16].

There are also a number of models in the mathematical physics literature that have this more complex asymptotic structure. In particular, Duplantier and Saleur [10] and Duplantier and David [9] studied the case of *dense* polymers in two dimensions, and found the partition functions had the asymptotic form (6). In [25], Owczarek, Prellberg and Brak investigated an exactly solvable model of interacting partially-directed self-avoiding walks (IPDSAW), for which the solution had previously been given by Brak, Guttmann and Whittington in [5]. In [25], Owczarek et al. analysed a 6000 term series expansion for IPDSAWs in the collapsed regime, and estimated $\sigma = 1/2$, $g = -3/4$, while μ_1 was estimated to at least 6 digit accuracy. From [5] the value of μ is exactly known. Subsequently Duplantier [8] pointed out that $\sigma = 1/2$ is to be expected for SAWs in the collapsed regime, for the two-dimensional version of these models. In all the examples we mention above, σ takes the value $1/3$ or $1/2$.

In the next section we give details of the enumeration algorithm. In subsequent sections we analyse the available series coefficients.

2. Algorithm

The algorithm used can be considered to be a set of further optimisations on the JN algorithm. However, a significantly different notation is used as this helps to make some of the optimisations clearer, as well as helping with a memory efficient encoding implementation. We note that Marinov and Rodoičić [24] have previously given a recursive algorithm for this problem. It is a significantly different algorithm conceptually to the JN algorithm, although one could imagine a variant on that algorithm keeping track of 12 patterns as part of their labels, instead of their set of $l(\pi)$ values. Such an algorithm would probably be similar in performance to this algorithm.

2.1. Basic algorithm

We will start with a very simple (and inefficient) algorithm. Let $f(n, P)$ be the number of permutations avoiding the pattern 1324 with n numbers remaining and starting with the prefix P (a sequence of integers). Then the desired series is $f(n, \emptyset)$. Each value $f(n, P)$ can be expressed as the sum of up to n other terms $f(n-1, P')$ where P' is P followed by one extra term. There will be fewer than n terms if P' implies a 1324 pattern, usually due to containing a 132 pattern with a 4 inevitably to eventually follow. Using the termination condition $f(0, -) = 1$, one could easily encode a recursive algorithm that would work.

This algorithm will systematically individually enumerate every permutation avoiding 1324, and its time consumption will be proportional to the answer. Like many such recursive enumeration algorithms, one can get a much faster algorithm by recognising that there exist many classes $S = (n, P_1, P_2, P_3, \dots)$ such that $f(n, P_i) = f(n, P_j)$ for all i and j . Define $f(S) = f(n, P_-)$, P_- means any prefix from the class. Now, modify the algorithm to use S (which will henceforth be called a signature) in a recursive function $f(S)$. After computing a value of $f(S)$, store it in some table. When you next need $f(S)$, look it up in the table. If it is already there, then use the stored value. This can vastly improve the speed of execution as you will avoid passing through large swathes of the enumeration tree. It does have a cost of memory. This approach is often called dynamic programming, memoisation, or memorisation.

When using such approaches, the definition of the signatures S is paramount. The more prefixes you can prove to be identical (and thus members of the same signature), the more efficient the algorithm will be. Indeed, the time and memory will both be proportional to the number of different signatures. The rest of this subsection deals with the definition of prefixes.

The signature must contain enough information to allow the algorithm to avoid 1324 patterns. One way to do this is to keep track of all the 13 patterns. Then, in the recurrence

Table 1
Example prefixes and corresponding signatures for $n = 20$.

Prefix	Signature	Explanation
\emptyset	20	20 numbers left to go
11	10,9	ten numbers to the left of the lowest number, 9 to the right
11,14	10,[2]6	a 13 pattern introduced with two numbers between it, and 6 to the right
11,14,5	4,5[2]6	a new lowest number, but no new 13 pairs
11,14,5,9	4,[3]1[2]6	a new 13 pair produced
11,14,5,9,15	4,[3]1[2]5	a new 13 pair produced

relation, you do not allow any numbers in the middle of one of these 13 patterns (i.e. a 2) if there are any numbers remaining bigger than the 3 (i.e. a 4, which would then inevitably follow at some point). There is no need to keep track of exactly what the numbers are; you just need to know the total number of numbers between, above, and below each 13 pattern.

One suitable notation to keep track of the 13 patterns is as a partition of integers (the number of numbers left to go) with a well-formed set of brackets (the 13 patterns). It is also necessary to keep track of the lowest number so far in the prefix, as all future numbers higher than it will form a 13 pattern with it. They may of course form other 13 patterns, but the one with the lowest 1 will be the most restrictive, and the others may be ignored. The lowest number is recorded with a comma. For brevity, the comma may be left out if there is a bracket immediately following it.

Some typical signatures are shown in Table 1, for some example prefixes and initial $n = 20$.

At this point one can simplify the signature. A signature of the form $a[b[c]d$ (that is, with two consecutive closing brackets) has two restrictions. The outer bracket means that you can't have anything in b or c until d is finished. The inner bracket means that you can't have anything in c until d is finished. The outer bracket is strictly more restrictive, so the inner one is redundant and may be removed. So $a[b[c]d = a[bc]d$. This simplification means that we will never have two closing brackets in succession. For instance, $2[3[4]5$, after removing the inner bracket, becomes $2[7]5$ rather than needing to record that the inner 7 was at one point broken unto a 3 and 4.

Adjacent integers not separated by brackets or a comma can be added together.

Any tail bracket at the end of the signature of course can be removed; the brackets are only restrictive if there are larger numbers possible.

At this point there is an isomorphism to the JN algorithm [21]. Their final functional form (last equation in Section 2) is $H_n^0(t; b_1, \dots, b_n; k)$. Here k encodes the position of the comma, and b_i encodes the location of the open bracket corresponding to a closing bracket at position i . Enumerating using these signatures produces an algorithm basically identical in performance to the JN algorithm. The further simplifications described below will improve performance.

Repeated open brackets can also be simplified. Consider a signature of the form $a[[b]c]d$. The outer bracket means you cannot have anything in b or c until everything in d is done. The inner bracket means you cannot have anything in b until c is done. This is

Table 2

All signatures used to compute up to the $n = 6$ term. Single lines are used when a higher n is started.

Signature S	Composed of	$f(S)$
1	,	1
,1	,	1
2	,1 + 1	2
,2	,1 + ,1	2
1,1	,1 + 1	2
3	,2 + 1,1 + 2	6
,3	,2 + [1]1 + ,2	5
1,2	,2 + 1,1 + 1,1	6
2,1	,2 + 1,1 + 2	6
4	,3 + 1,2 + 2,1 + 3	23
,4	,3 + [1]2 + [2]1 + ,3	14
1[1]1	[1]1 + 1,1	3
1,3	,3 + 1,2 + 1[1]1 + 1,2	20
2,2	,3 + 1,2 + 2,1 + 2,1	23
3,1	,3 + 1,2 + 2,1 + 3	23
5	,4 + 1,3 + 2,2 + 3,1 + 4	103
,5	,4 + [1]3 + [2]2 + [3]1 + ,4	42
1[1]2	[1]2 + 1[1]1 + 1[1]1	8
1[2]1	[2]1 + 1,2	8
1,4	,4 + 1,3 + 1[1]2 + 1[2]1 + 1,3	70
,1[1]1	[1]1 + ,2	3
2[1]1	,1[1]1 + 1[1]1 + 2,1	12
2,3	,4 + 1,3 + 2,2 + 2[1]1 + 2,2	92
3,2	,4 + 1,3 + 2,2 + 3,1 + 3,1	103
4,1	,4 + 1,3 + 2,2 + 3,1 + 4	103
6	,5 + 1,4 + 2,3 + 3,2 + 4,1 + 5	513

equivalent to the restrictions described by the signature $a[b][c]d$. Simplifying signatures by getting rid of all consecutive open brackets reduces the total number of signatures significantly, in practice by a factor of roughly 4.

A minor simplification comes from dealing with open brackets at the start of the signature. $[a]b$ means that everything in b must be dealt with before everything in a . This means that they are decoupled; indeed $f([a]b) = f(a)f(b)$. Factorising the problem seems like a big advantage, but only a small proportion of signatures starts with brackets; in practice this reduces the number of signatures by a factor of roughly 2.

An example of all the computations done (in the order that they are finished) for permutations of length 6 is given in [Table 2](#).

2.2. Other techniques to reduce memory consumption

A more complex simplification comes from noticing that a signature of the form $a[b]c$ will not touch any of the b values until all of c is dealt with. This means that there will be a set of signature prefixes p_i with multiplicities m_i such that $f(a[b]c) = \sum_i m_i f(p_i b)$. These p_i and m_i can be computed when needed and cached. If you did this for all $a[b]c$ you would spend more time and memory on this optimisation than the original problem

involved. However, if you just do it for sufficiently short a (in practice we used length of $a < 8$) then it can reduce memory consumption by about 30 percent without significant effect on speed. Note that the length of a is more important than the length of b , as all prefixes p_i must be no longer than a , so a small length of a ensures that the number of terms here does not get too large. Also note that $[b]$ can actually be a series of brackets, e.g. $[b_1][b_2[b_3][b_4]$.

Reducing the number of signatures reduces both the memory and the execution time, but memory consumption tends to be the bottleneck. One simple trick for problems of this class is to only save the result some fraction p of the time. Signatures that are only used once may not take up memory, and frequently used signatures will get stored eventually. The smaller p , the less memory used, but the more time used. We found that $p = 0.3$ reduced memory consumption by about 30 percent with about a 30 percent increase in execution time. This indicates that a significant fraction of the values $f(S)$ computed is only used once.

2.3. Implementation

The algorithm was implemented in Scala, which compiles to Java virtual machine bytecode, and run on a computer with 1 TB RAM.

The signature described here is less straightforward to encode on a computer than the array of 32 bytes in [21]. However, in practice it can be easily done in 128 bits for all the signatures needed for n up to the 50 s. This makes the keys 16 bytes, reducing memory consumption.

The signatures are encoded as a bitstring. The bitstring starts with a 6 bit number equal to the sum of the integers in the signature (n). This is used for determining when the bitstring stops. There is then one bit to indicate whether the signature starts with a comma. Then there is a repeating series of 1 bit for whether there is an open bracket, 2 to 8 bits for encoding an integer, and one bit for whether there is a closing bracket after that integer. This repeats until everything is encoded.

Integers were encoded as follows:

- 00 means the number 1,
- 01 means the number 2,
- 10bbb means the number $bbb + 3$,
- 11bbbbbb means the number $bbbbbb + 11$.

Some signatures have a large number of small numbers in them; this will encode them in a small number of bits. Others have a small number of large numbers; this will also encode them in a small number of bits. Indeed, up until $n = 29$, only 64 bits are needed for the keys.

The values are stored as 64 bit values *modulo* some large prime. The computation is redone *modulo* a different prime and the values reassembled using the Chinese

Remainder Theorem. This somewhat reduces memory use relative to 128 bit values, but also simplifies implementation as the java virtual machine does not handle 128 bit integers easily.

The main memoisation store was therefore effectively a large 128 bit key to 64 bit value hash table. As the upper 64 bits of the key were sparsely used (indeed only 131 different values for 35 terms), a significant memory saving was generated by having a master hash map from the upper 64 bits to a slave hash map. The slave hash map then only needed to use the lower 64 bits as keys. In practice the slave hash maps were still further subdivided in one more layer; this was due to java virtual machine limitations on array size restricting the length of a hash map. But the end result is the main memory store used 64 bit keys and values in a gnu trove hash map.

Various of the tricks used increased computation time to a matter of days; we made a parallelised version which worked about twice as fast as the single threaded version on a 4 core desktop pc for testing, but turned out slower on the 32 core production machine than the single threaded version, possibly because of memory coherency overheads. So it was not used. The parallelisation comes from having different threads compute different signatures. If, in computing $f(S)$, the value $f(s)$ for a subsignature s is not available, then a placeholder is inserted in $f(S)$, and s is added to a priority queue of signatures to evaluate. When a processor has nothing else to do, it takes the signature with the lowest n from the priority queue and evaluates it. When a signature is finished, it fills in all the placeholders for that signature. Taking the signature with the lowest n prevents the queue from growing exponentially with n .

With all these improvements implemented, the algorithm produced 5 further terms with comparable computing resources. The calculation had to be run twice, *modulo* two different primes, and the result reconstructed by the Chinese Remainder Theorem. As a check, we also ran it a third time, *modulo* another prime. The primes used were $2^{62} - k_i$, where $k_1 = 87$, $k_2 = 117$, $k_3 = 143$. For each of the two moduli used, the program ran for 5 days and used somewhat over 540 GB of memory. This is the amount of memory used at the termination of the algorithm as reported by the JVM; some extra is needed for various tasks, primarily temporary objects such as resizing of hash tables.

Source code is available on <https://github.com/AndrewConway/enumeration/> in the folder avoid1324.

The coefficients are given in Table 3.

3. Analysis

In the case of a simple algebraic singularity with asymptotic form (5), the ratio of the coefficients is

$$r_n = \frac{a_n}{a_{n-1}} \sim \mu \left(1 + \frac{g}{n} + O\left(\frac{1}{n^2}\right) \right). \quad (8)$$

Table 3
Coefficients of 1324 pattern-avoiding permutations.

1	3 421 888 118 907
1	25 887 131 596 018
2	198 244 731 603 623
6	1 535 346 218 316 422
23	12 015 325 816 028 313
103	94 944 352 095 728 825
513	757 046 484 552 152 932
2762	6 087 537 591 051 072 864
15 793	49 339 914 891 701 589 053
94 776	402 890 652 358 573 525 928
591 950	3 313 004 165 660 965 754 922
3 824 112	27 424 185 239 545 986 820 514
25 431 452	228 437 994 561 962 363 104 048
173 453 058	1 914 189 093 351 633 702 834 757
1 209 639 642	16 130 725 510 342 551 986 540 152
8 604 450 011	136 664 757 387 536 091 240 503 406
62 300 851 632	1 163 812 341 034 817 216 384 582 333
458 374 397 312	9 959 364 766 841 851 088 593 974 979
	85 626 551 244 475 524 038 311 935 717

If on the other hand the coefficients of some generating function are as in Eq. (6), then the ratio of successive coefficients $r_n = b_n/b_{n-1}$, is

$$r_n = \mu \left(1 + \frac{\sigma \log \mu_1}{n^{1-\sigma}} + \frac{g}{n} + \frac{\sigma^2 \log^2 \mu_1}{2n^{2-2\sigma}} + \frac{(\sigma - \sigma^2) \log \mu_1 + 2g\sigma \log \mu_1}{2n^{2-\sigma}} + \frac{\sigma^3 \log^3 \mu_1}{6n^{3-3\sigma}} + O(n^{2\sigma-3}) + O(n^{-2}) \right). \tag{9}$$

In particular, when $\sigma = \frac{1}{2}$, this specialises to

$$r_n = \mu \left(1 + \frac{\log \mu_1}{2\sqrt{n}} + \frac{g + \frac{1}{8} \log^2 \mu_1}{n} + \frac{\log^3 \mu_1 + (6 + 24g) \log \mu_1}{48n^{3/2}} + O(n^{-2}) \right). \tag{10}$$

3.1. Differential approximant analysis

The most successful numerical method for extracting the asymptotics from the first few terms of the OGF of a function with an algebraic singularity is the method of differential approximants (called the DA method), due to Guttmann and Joyce [18], with subsequent refinements due to Baker and Hunter [19] and Fisher and Au-Yang [11]. Details are given in [16,17,15]. In brief, the method fits available coefficients to a judiciously chosen family of D-finite ordinary differential equations (ODEs), and the singularity structure of the ODEs is extracted by standard methods [20,13].

For models with an isolated algebraic singularity, the method is very successful, with the radius of convergence and critical exponents frequently estimated to 10 significant digit accuracy or better, from a series of length 30–80 terms. However, when the method is used to analyse models with singularities that are not algebraic, such as those whose coefficients have the asymptotic form (6), the method fails, though in a predictable

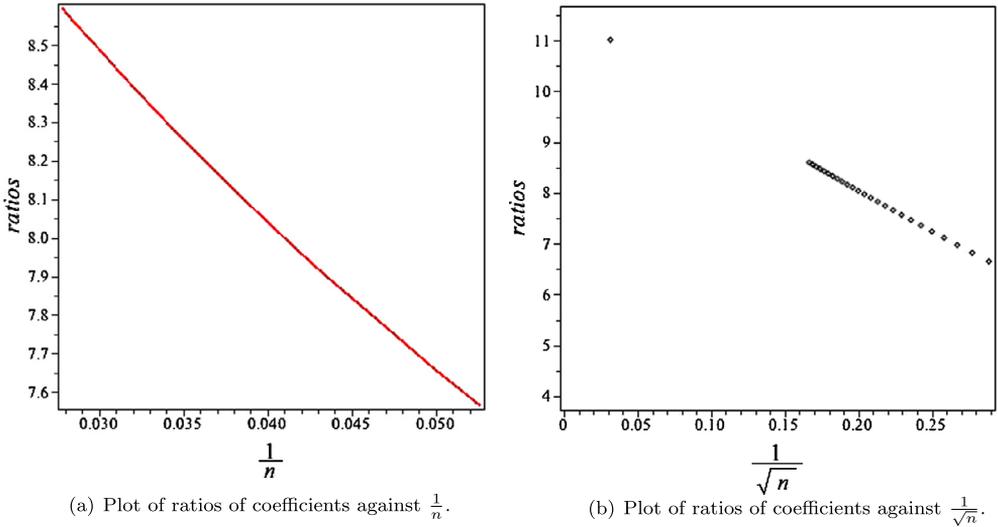


Fig. 1. Ratio plots against $1/n$ and against $1/\sqrt{n}$. Linearity occurs in the latter case.

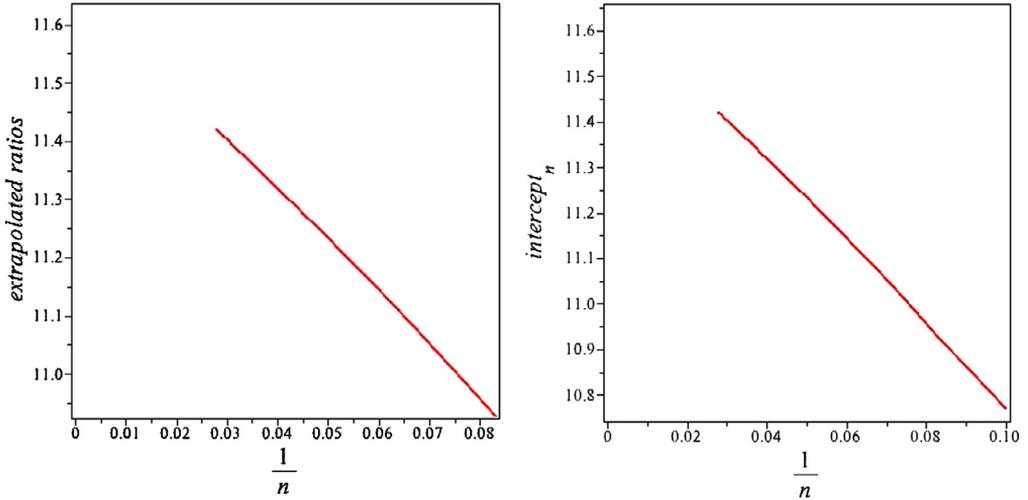
manner. That is to say, one finds that the radius of convergence estimates is typically only found to two or three significant digits, and the critical exponent estimates are numerically large, typically around 10 or -10 . This is discussed in greater detail in [16].

In this way, the method is useful – as is a canary in a coal mine. If one analyses the known terms of the series with the method of differential approximants and finds estimates of the radius of convergence to be poorly converged, with numerically large exponent values, one can be confident that the underlying OGF does not have an algebraic singularity. Applying the method to the first 30 terms of the 1342 and 1234 PAPs, the known solutions are found. Applying the method to the 36 coefficients we have for 1324 PAPs, the method suggests that the radius of convergence is around 0.09, with an exponent variously estimated to be -20 or $+15$ or anything in between! This is the hallmark of a non-algebraic singularity.

Further details of the DA method, its successes and limitations are discussed in [16]. For the moment, we simply conclude that the OGF of 1324 PAPs almost certainly does not have an algebraic singularity. In the next section we explore the nature of the singularity by looking at the ratio of successive coefficients.

3.2. Ratio analysis

In order to determine the nature of the asymptotic form of the coefficients of the 1324-PAP OGF, we first plot the ratios of successive coefficients $r_n = p_n/p_{n-1}$ against $1/n$, as shown in Fig. 1(a). The locus is clearly concave. This is inconsistent with an algebraic singularity, as can be seen from Eq. (8). We next plot the same ratios against $1/\sqrt{n}$ in Fig. 1(b), this time as a point plot, and the plot is seen to be visually linear, implying, from Eq. (9) that $\sigma \approx 1/2$. The outlying point near to the vertical axis is a



(a) Plot of extrapolated ratios against $\frac{1}{n}$. (b) Plot of estimators of μ by square root intercepts.

Fig. 2. Estimates of limiting ratio by extrapolation. The limit gives the growth constant.

Monte Carlo result obtained by Steingrímsson [27] for PAPs of length 1001. Madras and Liu [23] have previously pointed out the apparent linearity of the ratios when plotted against $1/\sqrt{n}$.

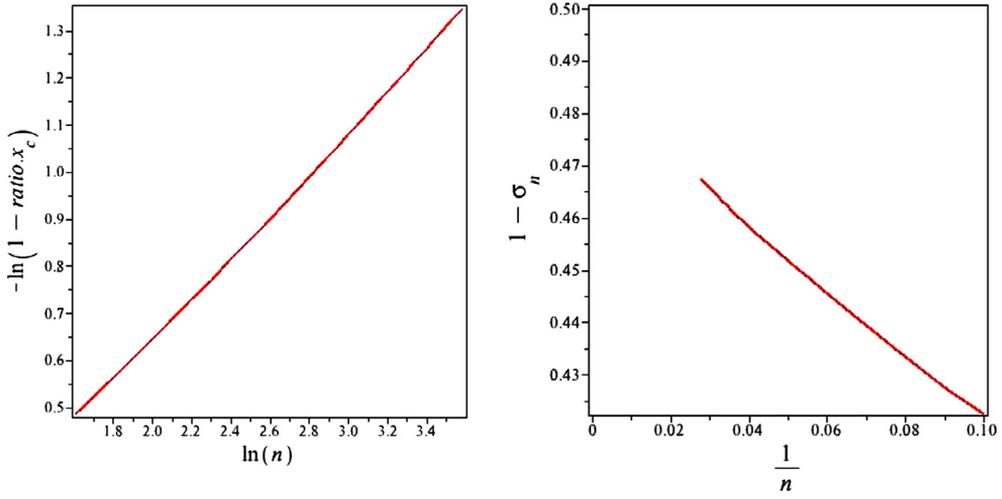
Linear extrapolation (not including the isolated point) implies a limiting value as $n \rightarrow \infty$ around 11.5. We can significantly improve on this estimate by considering the sequence of extrapolants defined by successive pairs of points. That is to say, one can simply linearly extrapolate successive pairs of ratios (r_k, r_{k+1}) with k increasing up to the maximum value achievable with our data, which is 35. A plot of successive extrapolants against $1/n$ is shown in Fig. 2(a), which appears to be linear. A crude extrapolation with a ruler suggests a limit of around 11.60.

Assuming (tentatively) that $\sigma = 1/2$, another way to use the ratios to get a better estimate of μ is to eliminate the assumed term $O(1/\sqrt{n})$ in the ratios by forming the modified ratios

$$intercept_n = \frac{\sqrt{n} \cdot r_n - \sqrt{n-1} \cdot r_{n-1}}{\sqrt{n} - \sqrt{n-1}} \sim \mu \cdot (1 + O(1/n)). \tag{11}$$

We show in Fig. 2(b) a plot of $intercept_n$ against $1/n$, and this too appears to be going to a value very close to 11.60. We will take that as our initial estimate, which we subsequently refine. We also take $\sigma = 1/2$ as our (initial) conjectured value. In doing so we are, in part, relying on the observation that in all known cases [16] where this asymptotic behaviour is observed, σ is a simple rational, usually $1/2$ or $1/3$.

We also plotted (but don't display) the ratios against $1/n^{2/3}$, which is appropriate if $\sigma = 1/3$. In that case the locus was convex, rather than concave. So on the basis of ratio plots alone, $\sigma \approx 1/2$ is our tentative estimate.



(a) Log-log plot of $(1 - r_n/\mu)$ against n .

(b) Plot of local extrapolants of figure at left, estimating $1 - \sigma$.

Fig. 3. Extrapolations to calculate the exponent σ . The first plot should be linear, the second gives estimators of exponent $1 - \sigma$.

In order to more accurately determine the value of the exponent σ , we note from (9) that

$$(r_n/\mu - 1) \sim const \cdot n^{\sigma-1}.$$

We show in Fig. 3(a) a log–log plot of $(1 - r_n/\mu)$ against n , where we have taken 11.60 as the (tentative) value of μ . This should be linear, with gradient $\sigma - 1$. A small degree of curvature is evident. Accordingly, we extrapolate the local ratios, defined as

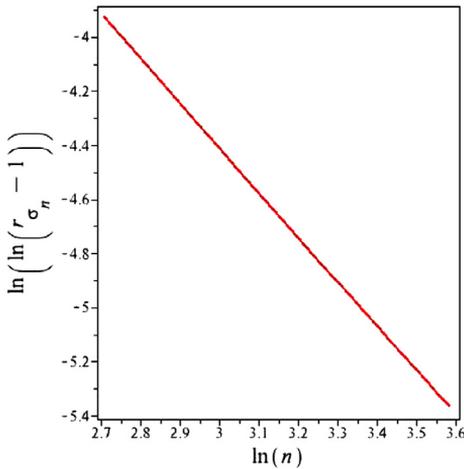
$$1 - \sigma_n = \frac{\log(1 - \frac{r_{n-1}}{\mu}) - \log(1 - \frac{r_n}{\mu})}{\log n - \log(n - 1)}, \tag{12}$$

against $1/n$. The results are shown in Fig. 3(b) The ordinates are estimators of $1 - \sigma$. If one accepts that σ is a simple rational number, the value $1/2$ is inescapable.

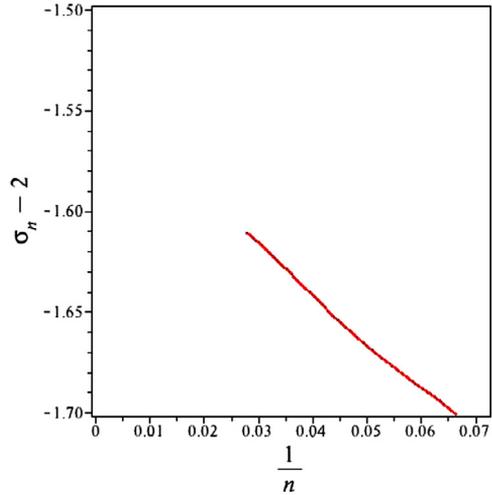
We can also estimate σ without assuming the value of μ as follows. From Eq. (6), one sees that

$$r_{\sigma_n} = \frac{b_n \cdot b_{n-2}}{b_{n-1}^2} \sim 1 + \frac{\sigma \cdot (\sigma - 1) \log \mu_1}{n^{2-\sigma}} + O(1/n^2), \tag{13}$$

so σ can be estimated from a log–log plot of $r_{\sigma_n} - 1$ against n , independent of the value of μ . From this plot, shown in Fig. 4(a), which is linear as expected, we calculate the local gradient at each pair of successive points, as described above, and plot these against $1/n$. This is shown in Fig. 4(b), and it can be seen that the expected limit, as $n \rightarrow 0$ is, plausibly, -1.5 . From Eq. (13), this limit should be $\sigma - 2$, which is consistent with our assertion that $\sigma = 1/2$. In our subsequent analysis, we will assume this value.



(a) Log-log plot of r_{σ_n} against n .



(b) Local gradient of log-log plots of r_{σ_n} against $\frac{1}{n}$.

Fig. 4. Alternative extrapolations to calculate the exponent σ . The first plot should be linear, the second gives estimators of exponent $\sigma - 2$.

Table 4

Last seven entries in each row of the table of Bulirsch–Stoer extrapolants. Each successive row is the result of a successively higher degree of extrapolation. The ratios of successive coefficients are extrapolated here, with parameter $w = 1/2$.

L	T(L,N-L-6)	T(L,N-L-5)	T(L,N-L-4)	T(L,N-L-3)	T(L,N-L-2)	T(L,N-L-1)	T(L,N-L)
1	13.1779207	13.1076887	13.0433300	12.9841336	12.9294976	12.8789107	12.8319353
2	11.6794491	11.6774670	11.6753818	11.6732009	11.6709428	11.6686318	11.6662949
3	11.6437016	11.6384623	11.6327588	11.6269378	11.6212795	11.6159990	11.6112494
4	11.7302686	11.7185072	11.7174627	11.7216794	11.7312898	11.7494782	11.7863032

With σ taken to be $1/2$, we attempt to refine the estimate of μ by extrapolating the ratios of the coefficients r_n using the Bulirsch–Stoer [6] algorithm, with parameter $w = 1/2$. This algorithm extrapolates a sequence $\{s_n\}$ assuming $s_n \sim s_\infty + c/n^w + o(n^{-w})$, where w is provided by the user. The results are given in Table 4. Each successive row represents a higher order of extrapolation. We only show the last 7 entries of each order of extrapolation. We continue generating rows until the row entries lose monotonicity. From the first row we conclude $\mu < 12.832$. From the second we conclude $\mu < 11.6663$, and from the third we conclude $\mu < 11.6112$. The rate of decrease in the entries in the third row is consistent with our initial estimate of 11.60, so we will retain this estimate for the time being, as the extrapolation has not given us a more precise value. It does however add support to that choice.

Assuming then that $\sigma = 1/2$, from (10), it follows that

$$r_n/\mu = 1 + \frac{\log \mu_1}{2\sqrt{n}} + \frac{g + \frac{1}{8} \log^2 \mu_1}{n} + O(n^{-3/2}).$$

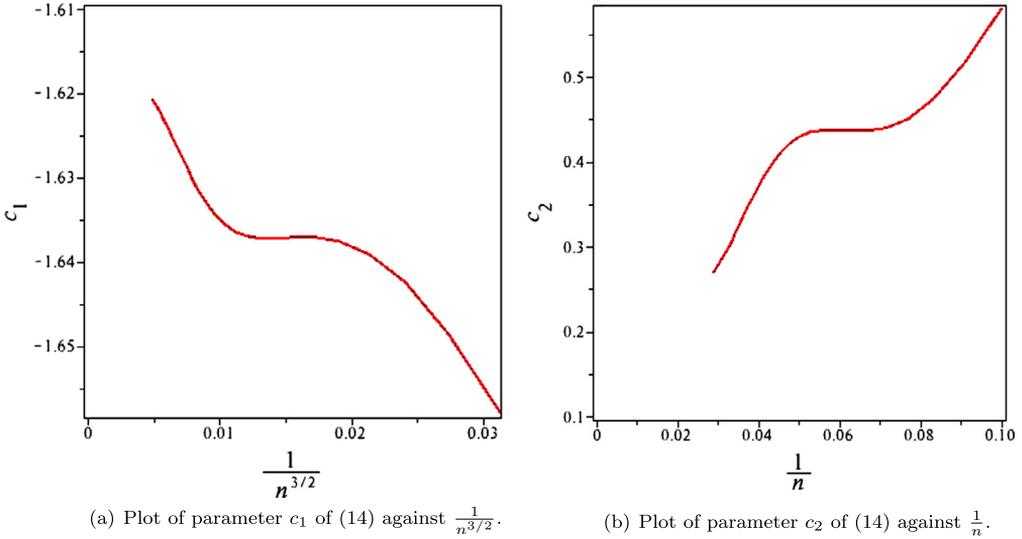


Fig. 5. Estimators of amplitudes c_1 and c_2 which yield values for growth constant μ_1 and exponent g .

In order to estimate μ_1 and g , we solve, sequentially, the trio of equations

$$r_j/\mu = 1 + \frac{c_1}{\sqrt{j}} + \frac{c_2}{j} + \frac{c_3}{j^{3/2}}, \tag{14}$$

for $j = k - 1$, $j = k$ and $j = k + 1$, with k ranging from 2 up to 35, and μ set at 11.60.

The results are shown in Figs. 5(a) and 5(b), plotting the parameters c_1 and c_2 respectively. The first neglected term in the asymptotics is $O(1/n^2)$ which is $O(1/n^{3/2})$ smaller than the term with coefficient c_1 , so c_1 is plotted against $1/n^{3/2}$. By a similar argument, c_2 is plotted against $1/n$. A simple visual extrapolation gives the estimate $c_1 = -1.615 \pm 0.005$. The plot for c_2 is difficult to extrapolate. It appears to be turning near its end point, and we very tentatively estimate $c_2 \approx 0.15$. Unless the gradient changes sign, we can only say that $c_2 < 0.2$, and it seems to be going to a positive value. If the gradient changes sign, we can't even say that. We don't show the plot of c_3 as we cannot extrapolate it. From (10), $c_1 = \log \mu_1/2$ and $c_2 = g + \frac{1}{8} \log^2 \mu_1$. Hence we estimate $\log \mu_1 \approx -3.23$, and assuming c_2 is in the range $[0, 0.2]$ this gives $g = -1.2 \pm 0.2$. We repeated this analysis varying the estimate of μ in the range $[10.58, 11.62]$. With μ in this range, we estimate $\log \mu_1 = 3.23 \pm 0.07$, and $g = -1.2 \pm 0.4$.

An alternative form of analysis involves direct fitting to the parameters in the assumed asymptotic form. That is to say, the assumed asymptotic form is

$$b_n \sim B \cdot \mu^n \cdot \mu_1^{n^\sigma} \cdot n^g.$$

Therefore

$$\log b_n \sim \log B + n \log \mu + n^\sigma \log \mu_1 + g \log n. \tag{15}$$

So if σ is known, or assumed, we have four unknowns in this linear equation. It is then straightforward to solve the linear system

$$\log b_k = c_1 k + c_2 k^\sigma + c_3 \log k + c_4$$

for $k = n - 2, n - 1, n, n + 1$ with n ranging from 3 to 35. Then c_1 estimates $\log \mu$, c_2 estimates $\log \mu_1$, c_3 estimates g and c_4 gives estimators of $\log B$.

An obvious useful variation is in those cases where, say, μ is known, or accurately estimated. Then one can solve

$$\log b_n - \mu \log n = c_1 n^\sigma + c_2 \log n + c_3 \tag{16}$$

from three successive coefficients b_{n-1}, b_n, b_{n+1} , as before increasing the order of the lowest used coefficient by one until one runs out of coefficients. We do this below with μ varying within its estimated error range.

Fitting the available coefficients to the four unknowns, we estimate $c_1 \approx 2.450 \pm 0.002$, implying $\mu = 11.59 \pm 0.02$ (in good agreement with our earlier estimate of 11.60), $c_2 = -3.23 \pm 0.03$, implying $\mu_1 = 0.0396 \pm 0.0012$, $c_3 \approx -1$, while c_4 is difficult to estimate beyond saying it is in the range $[1.1, 2.5]$ implying $B \in [3, 12]$.

We repeated this analysis with a 3 parameter fit, varying μ in the range $[11.58, 11.62]$. This gave $c_2 = -3.22 \pm 0.08$, implying $\mu_1 = 0.040 \pm 0.003$, $c_3 = g = -1.15 \pm 0.2$, and $c_4 = 1.7 \pm 0.6$ implying $B = 7 \pm 4$.

As noted above, differential approximants are useful insofar as they indicate that the singularity is not algebraic. They provide a signal, but are then of no further use in their current form. The presence of the $\mu_1^{\sqrt[n]{n}}$ term is responsible for the lack of applicability of the method. However, we can manipulate the series to remove the offending term, and then use this powerful method. From Eq. (15), defining $\tilde{b}_n = b_n/\sqrt[n]{n}$, one has

$$c_n = 2n^{3/2}(\tilde{b}_n - \tilde{b}_{n-1}) \sim (2g - \log B) + n \log(\mu) - g \log(n). \tag{17}$$

So

$$d_n = \exp(c_n) \sim D \cdot \mu^n \cdot n^{-g} \cdot (1 + O(n^{\sigma-1})),$$

with $D = \exp(2g)/B$.

The dominant asymptotics of the coefficients d_n are now those of an algebraic singularity – though the nature of the correction term, $O(n^{1-\sigma})$, means that there is a confluent singularity with exponent less than 1. This means that the OGF $\sum d_n \cdot x^n$ can be analysed by standard methods used to analyse series with algebraic singularities. This includes the method of differential approximants and Bulirsch–Stoer extrapolation of ratios. For the former method, one should use third-order DAs, as the presence of a confluent singularity means that we require an ODE with two independent solutions, and to allow for a non-singular background term requires a third independent solution.

Table 5

Critical point and exponent estimates for renormalised 1324 PAPs.

L	Second-order DA		Third-order DA	
	$1/\mu$	$g - 1$	$1/\mu$	$g - 1$
0	0.086237	-2.160	0.086134	-1.925
1	0.086116	-1.905	0.086156	-1.958
2	0.086142	-1.942	0.086149	-1.944
3	0.086159	-1.964	0.086167	-1.982
4	0.086137	-1.926	0.086156	-1.960
5	0.086111	-1.914	0.086162	-1.966
6	0.086110	-1.910	0.086158	-1.959
7	0.086140	-1.926	0.086170	-1.980
8	0.086143	-1.936	0.086178	-1.997
9	0.086145	-1.934	0.086186	-2.012
10	0.086142	-1.931	0.086188	-2.011

Table 6

Last seven entries in each row of the table of Bulirsch–Stoer extrapolants. Each successive row is the result of a successively higher degree of extrapolation. The ratios of successive coefficients of the transformed series are extrapolated here.

L	T(L,N-L-6)	T(L,N-L-5)	T(L,N-L-4)	T(L,N-L-3)	T(L,N-L-2)	T(L,N-L-1)	T(L,N-L)
1	11.6352643	11.6360798	11.6364819	11.6365193	11.6362465	11.6357199	11.6349941
2	11.6520073	11.6474449	11.6423878	11.6370974	11.6318101	11.6267299	11.6220231
3	11.6410222	11.6387019	11.6372697	11.6365437	11.6364700	11.6370951	11.6385979

For Bulirsch–Stoer extrapolation of ratios, the parameter $w = 1$ should be used, as the ratios $d_n/d_{n-1} \sim \mu \cdot (1 - g/n + o(1/n))$.

For the DA analysis we have used third- and fourth-order ODEs. As an aside, we also tried second-order ODEs and these were unsatisfactory, as we expected, as they produced two singularities close together in an unsuccessful attempt at representing the confluent singularity. We summarise the results in [Table 5](#). The column labelled L gives the degree of the inhomogeneous polynomial of the approximating ODEs. The entries give estimates, averaged over many approximants, of the position and exponent of the singularity of the ODEs. Full details of the method are given in [\[15,17\]](#). It is seen that the third-order DAs give estimates of the radius of convergence that are centred around $0.086140 \approx 1/11.609$, with exponent estimates around $g \approx -0.93$. The fourth-order approximants give slightly higher estimates of both the critical point and the absolute value of the exponent. We estimate $1/\mu \approx 0.08619$, or $\mu \approx 11.602$. This is remarkably close to the initial estimate above, $\mu = 11.60$. For the exponent g we note that the estimates are decreasing with L , and estimate $g = -1.1 \pm 0.2$. This is the most stable estimate of this exponent, so we take this as our final estimate.

We can also apply other standard techniques to the transformed series. The ratios of successive terms (d_n) of the transformed series when plotted against $1/n$ are now visually linear. Accordingly, we extrapolate the ratios of the coefficients of the transformed series using the Bulirsch–Stoer algorithm, with parameter $w = 1$. The results are shown in [Table 6](#). The first two rows are behaving monotonically. The last entry in the second row suggests that $\mu < 11.622$, and assessing the way entries in that row are decreasing, we

Table 7

Estimates of the sub-dominant terms and amplitude for different values of the dominant term.

μ	$\log \mu_1$	Exponent g	Amplitude B
11.59	-3.18	-1.23	8.3
11.60	-3.21	-1.18	8.0
11.61	-3.25	-1.05	5.8

judge the limit to be around 11.60 ± 0.01 . Together with the result of the DA analysis given above, we combine these two results and give as our final estimate $\mu = 11.60 \pm 0.01$. The quoted error is to be interpreted as a confidence interval, not a rigorous error bound. And, it should be stressed, our analysis is predicated on our assumption that $\sigma = 1/2$.

The amplitude B was estimated by extrapolating the sequence $b_n/(\mu^n \cdot \mu_1^{\sqrt{n}} \cdot n^g)$, using the estimates of the critical parameters already found, against $1/n$. However, the quoted parameter errors are correlated, and we take this into account in the analysis below.

The major difficulty in a more precise analysis is that the asymptotic form of the coefficients has three different scales. The dominant term is the exponential term μ^n . The first sub-dominant term is $\mu_1^{n^\sigma}$, where we have argued that $\sigma = 1/2$. Finally, the algebraic term n^g is the weakest asymptotic term. We have observed that the estimates of the sub-dominant terms depend significantly on the value of the dominant, exponential growth term.

Accordingly, we have reanalysed the data, varying the dominant exponential term within its range of uncertainty to more precisely estimate each sub-dominant term in turn. That is, we fix μ , then analyse for μ_1 , assuming $\sigma = 1/2$. We use several methods of analysis of the type discussed above. Having estimated μ_1 , we then analyse for g , with μ , μ_1 and σ given. Finally, we estimate the amplitude B , given the values of all the other parameters.

The results are shown in [Table 7](#).

4. Conclusion

We have given a refined version of the JN algorithm that allows five further coefficients of the 1324 PAP generating function to be obtained, with comparable computer resources.

Analysing the coefficients of the generating function, we provide compelling evidence that they have a singularity structure of the form

$$B \cdot \mu^n \cdot \mu_1^{n^\sigma} \cdot n^g.$$

We give as our final estimates of the critical parameters $\mu = 11.60 \pm 0.01$, $\sigma = 1/2$, $\mu_1 = 0.040 \pm 0.0015$, $g = -1.1 \pm 0.2$ and $B = 7 \pm 1.3$. If, as is seen in other problems which have coefficients with a similar asymptotic structure, g is a simple rational fraction, the most likely value is $-9/8$ or -1 , but we could not rule out any other rational close to

these values. Our estimate of the amplitude B is very sensitive to the other parameters, which is why it has a rather large associated uncertainty. Finally, we must stress that these estimates are dependent on our initial conjecture that $\sigma = 1/2$. If in fact $\sigma = 15/32$ say, we would have difficulty distinguishing this from $1/2$. That does not however change our primary conclusion as to the general asymptotic form of the coefficients.

Taking the values of each parameter corresponding to the central estimate of μ from [Table 7](#), the coefficients are predicted to behave asymptotically as

$$8 \cdot 11.6^n \cdot 0.040\sqrt{n}/n^{1.18}.$$

Taking the ratio of this estimate to the actual coefficients for $n = 10, 20, 30, 36$ gives 0.67, 0.78, 0.84, 0.86 respectively, showing the slow approach to asymptopia. For $n = 1001$, Steingrímsson [\[27\]](#) gives the Monte Carlo estimate for the ratio b_n/b_{n-1} of 11.01146. The asymptotic form given above gives for this ratio the estimate 11.01154, in remarkable agreement.

Zeilberger has said “Not even God knows the number of 1324-avoiders of length 1000”. While making no Messianic claims, our asymptotics permit the approximate answer $5 \pm 1 \times 10^{1017}$.

Acknowledgments

A.J.G. wishes to acknowledge helpful conversations with Einar Steingrímsson, who brought this problem to our attention, and the hospitality of Mathematisches Forschungsinstitut Oberwolfach and the Enumerative Combinatorics Workshop held there on March 2–8, 2014, where this work was initiated. We also thank Neal Madras for helpful discussions and some unpublished Monte Carlo data. We are grateful to Alan Sokal, who gave us access to his Dell computers with 1 TB memory, which were needed for these calculations, through his NSF grant PHY-0424082 NYU. A.J.G. wishes to thank the Australian Research Council for supporting this work through grant DP120100931.

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