

NUMERICAL STUDIES OF THOMPSON'S GROUP F AND RELATED GROUPS

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ABSTRACT. We have developed polynomial-time algorithms to generate terms of the cogrowth series for groups $\mathbb{Z} \wr \mathbb{Z}$, the lamplighter group, $(\mathbb{Z} \wr \mathbb{Z}) \wr \mathbb{Z}$ and the Navas-Brin group B . We have also given an improved algorithm for the coefficients of Thompson's group F , giving 32 terms of the cogrowth series. We develop numerical techniques to extract the asymptotics of these various cogrowth series. We present improved rigorous lower bounds on the growth-rate of the cogrowth series for Thompson's group F using the method from [18] applied to our extended series. We also generalise their method by showing that it applies to loops on any locally finite graph. Unfortunately, lower bounds less than 16 do not help in determining amenability.

Again for Thompson's group F we prove that, if the group is amenable, there cannot be a sub-dominant stretched exponential term in the asymptotics¹. Yet the numerical data provides compelling evidence for the presence of such a term. This observation suggests a potential path to a proof of non-amenability: If the universality class of the cogrowth sequence can be determined rigorously, it will likely prove non-amenability.

We estimate the asymptotics of the cogrowth coefficients of F to be

$$c_n \sim c \cdot \mu^n \cdot \kappa^{n^\sigma \log^\delta n} \cdot n^g,$$

where $\mu \approx 15$, $\kappa \approx 1/e$, $\sigma \approx 1/2$, $\delta \approx 1/2$, and $g \approx -1$. The growth constant μ must be 16 for amenability. These two approaches, plus a third based on extrapolating lower bounds, support the conjecture [7, 18] that the group is not amenable.

1. INTRODUCTION

In an attempt to find compelling evidence for the amenability or otherwise of Thompson's group F , we have studied, numerically, the co-growth sequence of a number of infinite, finitely generated amenable groups whose asymptotics are, in most cases, partially or fully known. We have chosen a number of examples with increasingly complex asymptotics. Using the experience and insights gained from these examples, we turn to a study of Thompson's group F , having first developed an improved algorithm for the generation of the co-growth sequence, which we evaluate to $O(x^{32})$.

The cogrowth series of a group \mathcal{G} with finite, inverse closed, generating set S is

$$C_{\mathcal{G}} = \sum_{n \geq 0} c_n x^n,$$

where c_n is the number of words w of length $2n$ over the alphabet S , which satisfy $w =_{\mathcal{G}} 1$ i.e. w is the identity in the group \mathcal{G} . There are many equivalent definitions of amenability. A standard one is that a group G is amenable if it admits a left-invariant finitely additive

probability measure μ . A consequence of the Grigorchuk-Cohen [11, 5] theorem is that G is amenable if and only if the radius of convergence of C_G is $1/|S|^2$. In particular, Thompson's group F amenable if and only if its cogrowth sequence has exponential growth rate 16.

We have developed new, polynomial-time algorithms to generate coefficients for the lamplighter group, and for general wreath product groups, $W_d = \mathbb{Z} \wr_d \mathbb{Z}$. We also give a polynomial time algorithm for the cogrowth coefficients of the Navas-Brin group, and an improved algorithm to generate the coefficients of Thompson's group F , generating the cogrowth sequences to $O(x^{128})$ and $O(x^{32})$ for B and F respectively.

The amenable group introduced independently by Navas [19] and Brin [4], which we call the Navas-Brin group B , is a subgroup of Thompson's group F , and is defined as an infinite wreath product, with an extra generator which commutes each generator of the infinite wreath product to the next one. It has 2 generators, so the growth rate of the cogrowth sequence is 16. It also has a sub-exponential growth term that is very close to exponential, and so makes the growth rate difficult to estimate accurately with the number of terms at our disposal.

Using results of Pittet and Saloff-Coste [20, 21], we prove that the cogrowth coefficients c_n of Thompson's group F satisfy

$$c_n < 16^n \cdot \lambda^{-n^\kappa}$$

for any real numbers $\kappa < 1$, and $\lambda > 1$. That is to say, if Thompson's group F is amenable, then its asymptotics cannot contain a stretched-exponential term¹. Such a term is present in the asymptotics of the lamplighter group L and the family of groups W_d . Furthermore, our numerical study reveals compelling evidence for the *presence* of such a term in the asymptotics of the coefficients of F . This is our first strong evidence that Thompson's group F is not amenable. Our second piece of evidence is the estimation of the growth constant. For amenability, the growth constant must be 16. We find that it is very close to 15.0 (we do not suggest it is exactly 15, but that is certainly a possibility).

Our numerical analysis relies on a number of methods that are well-known in the statistical mechanics and enumerative combinatorics community. Many are reviewed in [13] and [16]. For studies of the cogrowth asymptotics we primarily rely on the behaviour of the *ratio* of successive coefficients, as irrespective of the sub-dominant asymptotics, this ratio must go to the growth constant in the limit as the order of the coefficients goes to infinity.

One new technique that we make use of in our study of the groups B and F is that of *series extension* [15]. In the case of group B , we have 128 exact coefficients, but predict a further 590 ratios (and terms) with an estimated accuracy of, at worst, 1 part in 5×10^{-7} . Having these extra (approximate) terms greatly improves the quality of the analysis we can perform. Similarly, for group F , we use 32 exact terms to predict a further 200 ratios

¹We define *stretched exponential* more broadly than usual. It normally refers to a term of the form e^{-t^β} , with $t > 0$ and $0 < \beta < 1$. We allow behaviour such as $e^{-t^\beta \cdot \log^\delta t}$, or indeed any appropriate logarithmic term. We do not have a name for sub-exponential growth of the form $e^{-t/\log^\delta t}$, with $\delta > 0$ (or appropriate logarithmic function) which is the type of term that must be present in the cogrowth series of the Navas-Brin group, and indeed in Thompson's group F if it were amenable.

(and terms) with an estimated accuracy of 1 part in 4×10^{-5} . This level of accuracy is more than sufficient for the graphical techniques we use to extract the asymptotics.

Another approach to estimating the growth rate was introduced by Haagerup, Haagerup and Ramirez-Solano in [18] who proved that the cogrowth sequence of Thompson's group F is given by the moments of a probability measure. We extend this to prove that this observation applies to the cogrowth sequence of any Cayley graph. In this way a sequence of rigorous lower bounds to the growth constant of the cogrowth series can be constructed. This approach also gives some stronger, non-rigorous, pseudo-bounds. Further details of this method, and some results, are given in section 4.

The simplest examples of groups we have chosen have asymptotics of the form

$$c_n \sim c \cdot \mu^n \cdot n^g,$$

where c is a constant, μ is the *growth constant* and g is an exponent.

The first example of such a group is \mathbb{Z}^2 , which is a particularly simple case as both the coefficients and generating function are exactly known. In fact $c_n = \binom{2n}{n}^2$, and the generating function $C_{\mathbb{Z}^2} = 2\mathbf{K}\left(\frac{4\sqrt{x}}{\pi}\right)$, where \mathbf{K} is the complete elliptic integral of the first kind.

The second example is the Heisenberg group, for which the asymptotic form of the coefficients is known [10] to be $c_n \sim 0.5 \cdot 16^n \cdot n^{-2}$, corresponding to a generating function

$$C_{\text{Heisenberg}} \sim \frac{1}{2}(1 - 16x) \log(1 - 16x).$$

We have calculated 90 terms of the generating function, and show that this is sufficient to get a very precise asymptotic representation of the coefficients.

The next level of asymptotic complexity arises when there is an additional stretched-exponential term, so that the coefficients of the generating function behave as

$$c_n \sim c \cdot \mu^n \cdot \kappa^{n^\sigma} \cdot n^g,$$

where $0 < \kappa < 1$, and $0 < \sigma < 1$. There is no known simple expression for the corresponding generating function in such cases². The lamplighter group L is the wreath product of the group of order two with the integers, $L = \mathbb{Z}_2 \wr \mathbb{Z}$. The growth rate is known, $\mu = 9$, and from Theorem 3.5 of [21] it follows that $\sigma = 1/3$, and from [22] we know that the exponent $g = 1/6$. So for the lamplighter group, $c_n \sim c \cdot 9^n \cdot \kappa^{n^{1/3}} \cdot n^{1/6}$. Methods to extract the asymptotics from the coefficients have been developed, and are described in [14]. We give a polynomial time algorithm to generate the coefficients, and use it to determine the first 201 coefficients, from which we are able to estimate the correct values of the parameters μ , σ and g .

We next consider wreath products $W_d = \mathbb{Z} \wr_d \mathbb{Z}$. In that case the exponent of the stretched-exponential term also includes a fractional power of a logarithm. Coefficients of the generating function behave as given by Theorem 3.11 in [21], so that

$$c_n \sim c \cdot \mu^n \cdot \kappa^{n^\sigma \log^\delta n} \cdot n^g,$$

²See, for example [14] for a discussion of this point, and further examples of such generating functions.

where $0 < \kappa < 1$, and $0 < \sigma, \delta < 1$.

For $d = 1$, one has $\mu = 16$, $\sigma = 1/3$, $\delta = 2/3$ and g is not known. For $d = 2$, one has, again by Theorem 3.11 in [21], $\mu = 36$, $\sigma = 1/2$ and $\delta = 1/2$. For general d , $\mu = (2d)^2$, $\sigma = d/(d+2)$, and $\delta = 2/(d+2)$.

Note that this dimensional dependence of the exponent σ of the stretched-exponential term appears to be a common feature among a broad class of problems. For example, if one considers the problem of a self-avoiding walk attached to a surface at its origin (or a Dyck path or a Motzkin path) and pushed toward the surface at its end-point (or its highest vertex), then, as shown in [1] there is a stretched-exponential term in the asymptotics of the coefficients, with exponent $\sigma = 1/(1 + d_f)$, where d_f is the fractal dimension of the walk/path. Whether this dimensional dependence is in fact a ubiquitous feature of such stretched-exponential terms remains an open question.

We have studied two examples, $W_1 = \mathbb{Z} \wr \mathbb{Z}$ and $W_2 = (\mathbb{Z} \wr \mathbb{Z}) \wr \mathbb{Z}$, based on the series we have generated of 276 and 133 terms respectively. We find that the presence of the confluent logarithmic term in the exponent makes the analysis significantly more difficult, but we can nevertheless accurately estimate the growth constant μ and less precisely estimate the sub-dominant growth rate κ and the exponents σ and δ . Our estimates of the exponent g are not precise enough to be useful.

We then turn to a contrived example, a constructed series with the asymptotics of $W_d = \mathbb{Z} \wr_d \mathbb{Z}$, with $d = 98$. As d increases, the exponent in the stretched-exponential term gets closer to 1, and so this term behaves more and more like the dominant exponential growth term μ^n . We show that estimating the correct growth constant even approximately requires careful analysis, and appropriate techniques. This serves as a caution, and underlies that our conclusions regarding the non-amenability of Thompson's group F assumes the absence of some unknown functional pathology.

Finally we study two groups whose behaviour is not fully known. The first is the Navas-Brin group B . We give a polynomial-time algorithm to generate the coefficients, and in this way generate the first 128 terms, then use these to estimate the next 590 ratios. This group has a sub-exponential growth term that is very close to exponential, and so makes the growth rate difficult to estimate accurately with the number of terms at our disposal. The second is Thompson's group F where we have 32 exactly known terms, and 200 estimated ratios of terms.

The makeup of the paper is as follows. In Section 2 we describe the algorithms developed for the cogrowth series of the lamplighter group L , W_1 , W_2 , B and Thompson's group F . In Section 3 we discuss the possible asymptotic form of the cogrowth series for Thompson's group F , and prove the absence of a stretched-exponential term. In Section 4 we develop the idea that the cogrowth coefficients can be represented as the sequence of moments of a probability measure. With this identification we establish rigorous lower-bounds on the growth constant for Thompson's group F . In Section 5 we analyse the series expansions for the cogrowth series of all the groups we have mentioned above, apart from B and F . Section 6 is devoted to a description of the method of series extension that we employ, and in Sections 7 and 8 we use this method and the techniques discussed in the previous

section to analyse the Navas-Brin group B and Thompson's group F . Section 9 comprises a discussion and conclusion.

2. SERIES GENERATION

In this section we describe the algorithms we have used to compute the terms of the cogrowth sequence of various groups. We start by describing polynomial time algorithms which we have found and used for the groups L , W_1 , W_2 , and B . Finally we describe the algorithm which we have used for Thompson's group F . The first 50 coefficients for the group B are given in Table 1, while the coefficients of the cogrowth series of F are given in Table 2.

2.1. Wreath Products $G \wr \mathbb{Z}$. Let G be a group with finite generating set S . We will describe a polynomial time algorithm for computing the cogrowth series of $G \wr \mathbb{Z}$, with respect to the generating set $\{a\} \cup S$, where a generates \mathbb{Z} , given the corresponding series for G . In particular, this give a polynomial time algorithm to compute the cogrowth of the lamplighter group $\mathbb{Z}_2 \wr \mathbb{Z}$ as well as groups such as $\mathbb{Z} \wr \mathbb{Z}$ and $(\mathbb{Z} \wr \mathbb{Z}) \wr \mathbb{Z}$.

Let a_k be the number of loops of length k in G . For example, if $G = \mathbb{Z}$, then $a_{2k} = \binom{2k}{k}$ and $a_{2k+1} = 0$ for all $k \in \mathbb{Z}_{\geq 0}$. Then for each positive integer n , define the generating function $P_n(x)$ by

$$P_n(x) = \sum_{j=0}^{\infty} \binom{j+n-1}{n-1} a_j x^j.$$

This is the generating function for n -tuples of words $w_1, w_2, \dots, w_n \in S^*$ such that $\overline{w_1 \dots w_n} = 1$, counted by the length of the word $w_1 \dots w_n$.

Given a loop l in $G \wr \mathbb{Z}$, we define the base loop l' of l to be the loop in \mathbb{Z} made up of only the terms a and a^{-1} in l . For each positive integer i , let c_i be the number of steps in the baseloop l' from a^{i-1} to a^i (which is the same as the number of steps from a^i to a^{i-1}) and let d_i be the number of steps from a^{-i+1} to a^{-i} . Let m and n be maximal such that $c_m, d_n > 0$. Then the length of l' is equal to

$$\sum_{i=1}^m 2c_i + \sum_{j=1}^n 2d_j.$$

Let $l' = a_1 a_2 \dots a_{|l'|}$ and $l = w_1 a_1 w_2 \dots a_{|l'|} w_{|l'|+1}$, where each w_i is a word in $(S \cup S^{-1})^*$. We say that the height of one of the subwords w_i is equal to the integer p which satisfies $a^p = a_1 \dots a_i$. Then l is a loop if and only if for any height h , concatenating all of the words w_i at height h creates a loop in G . Hence the generating function for the sections at height h is $P_r(x)$ where r is the number of these sections. If $h > 0$ then $r = c_h + c_{h+1}$, if $h < 0$ then $r = d_{-h} + d_{-h+1}$ and if $h = 0$ then $r = c_1 + d_1 + 1$. Hence, by considering the sections of l at each height separately, we see that the generating function for loops l with base loop l' is equal to

$$(1) \quad x^{|l'|} P_{d_n}(x) P_{c_m}(x) P_{c_1+d_1+1}(x) \prod_{i=1}^{m-1} P_{c_i+c_{i+1}}(x) \prod_{j=1}^{n-1} P_{d_j+d_{j+1}}(x),$$

assuming that $m, n \geq 1$. Similarly, if $m = 0$ and $n \geq 1$, the generating function is

$$x^{|l'|} P_{d_n}(x) P_{d_{1+1}}(x) \prod_{j=1}^{n-1} P_{d_j+d_{j+1}}(x).$$

If $n = 0$ and $m \geq 1$, the generating function is

$$x^{|l'|} P_{c_m}(x) P_{c_{1+1}}(x) \prod_{i=1}^{m-1} P_{c_i+c_{i+1}}(x).$$

Finally, if $m = 0$ and $n = 0$, then the generating function is $P_1(x)$. So we now need to sum this over all possible base loops l' .

For a given pair of sequences $c_1, \dots, c_m, d_1, \dots, d_n$, the number of such base loops is equal to

$$(2) \quad \binom{c_1 + d_1}{c_1} \prod_{i=1}^{m-1} \binom{c_i + c_{i+1} - 1}{c_i - 1} \prod_{j=1}^{n-1} \binom{d_j + d_{j+1} - 1}{d_j - 1}.$$

This is because from each vertex $i > 0$ we can choose the order of the outgoing steps, except that the last one must be a left step, and there are $c_i - 1$ other left steps and c_{i+1} right steps. Hence there are $\binom{c_i+c_{i+1}-1}{c_i-1}$ possible orders of the steps leaving any vertex $i > 0$, and similarly $\binom{d_j+d_{j+1}-1}{d_j-1}$ possible orders of the steps leaving any vertex $-j$ for $j > 0$. Finally, there are $\binom{c_1+d_1}{c_1}$ possible orders of the steps leaving the vertex 0. It is easy to see that for any possible choice of these orders there is exactly one corresponding base loop l' .

Now using (1) and (7) it follows that for any pair of sequences $c_1, \dots, c_m, d_1, \dots, d_n$, with $m, n \geq 1$, the generating function for the corresponding loops l in $G \wr \mathbb{Z}$ is equal to

$$(3) \quad x^{2c_1+2d_1} \binom{c_1 + d_1}{c_1} P_{d_n} P_{c_m} P_{c_{1+d_1+1}} \prod_{i=1}^{m-1} x^{2c_{i+1}} \binom{c_i + c_{i+1} - 1}{c_i - 1} P_{c_i+c_{i+1}} \prod_{j=1}^{n-1} x^{2d_{j+1}} \binom{d_j + d_{j+1} - 1}{d_j - 1} P_{d_j+d_{j+1}}.$$

If $m = 0$ and $n \geq 1$, the generating function is

$$(4) \quad x^{2d_1} P_{d_n} P_{d_{1+1}} \prod_{j=1}^{n-1} x^{2d_{j+1}} \binom{d_j + d_{j+1} - 1}{d_j - 1} P_{d_j+d_{j+1}}.$$

If $m \geq 1$ and $n = 0$ we get a similar generating function, and if $m = n = 0$ we get $P_1(x)$.

To calculate these we define some new power series $\Omega_d(x)$ by

$$\Omega_d(x) = \sum P_{d_n} \prod_{j=1}^{n-1} x^{2d_{j+1}} \binom{d_j + d_{j+1} - 1}{d_j - 1} P_{d_j+d_{j+1}}(x),$$

where the sum is over all sequences n, d_1, d_2, \dots, d_n with $d_1 = d$. Then it follows immediately from (3) and (4) that the generating function F for the cogrowth series series of $G \wr \mathbb{Z}$

is given by

$$(5) \quad F(x) = \left(\sum_{c,d=1}^{\infty} x^{2c+2d} \binom{c+d}{c} P_{c+d+1}(x) \Omega_d(x) \Omega_c(x) \right) + 2 \left(\sum_{d=1}^{\infty} x^{2d} P_d(x) \Omega_d(x) \right) + P_1(x).$$

So now we just need to calculate $\Omega_d(x)$ for each positive integer d . First, the contribution to Ω_d from the case where $n = 1$ is $P_{d_n} = P_{d_1} = P_d$. The contribution from the case where $n = 1$ and $d_2 = b$ for some fixed positive integer b is

$$x^{2b} \binom{d+b-1}{d-1} P_{d+b}(x) \Omega_b(x).$$

Hence, we have the equation

$$(6) \quad \Omega_d(x) = P_d(x) + \sum_{b=1}^{\infty} x^{2b} \binom{b+d-1}{d-1} P_{b+d}(x) \Omega_b(x).$$

Using this equation we can calculate the coefficient of x^k in Ω_d of x in terms of coefficients of x^j in $\Omega_b(x)$ where we only need to consider j, b satisfying $2b + j \leq k$ (hence $j \leq k - 2$). This takes polynomial time using a simple dynamic program.

2.2. The Navas-Brin group B . In this section we adapt the previous algorithm to calculate the cogrowth series for the Navas-Brin group B . Again this is a polynomial time algorithm, however the polynomial has higher degree than the one for the previous section. The group B is defined as the semi-direct product

$$(\dots \wr \mathbb{Z} \wr \mathbb{Z} \wr \mathbb{Z} \wr \mathbb{Z} \wr \dots) \rtimes \mathbb{Z},$$

where the copies of \mathbb{Z} in the wreath product are generated by $\dots, a_2, a_1, a_0, a_{-1}, a_{-2}, \dots$ and the generator t of the other copy of \mathbb{Z} satisfies $ta_it^{-1} = a_{i+1}$ for each i . Note that the group B is generated by the two elements t and $a = a_0$. The group B was described independently in [19] and on page 638 in [4], where Brin showed that is an amenable supgroup of Thompson's group F . In that paper it is the group generated by f and h .

We define the t -height of a word over the generating set $\{a, t, a^{-1}, t^{-1}\}$ to be the sum of the powers of t . Before counting the total number of loops, we will count the number of loops where any initial subword has non-negative height. Let $G(x, y)$ be the generating function for these, where x counts the total length and y counts the number of steps of the loop which end at height 0. For each positive integer n , let $H_n(x, y)$ be the generating function for n -tuples w_1, w_2, \dots, w_n of words in $\{a, a^{-1}, t, t^{-1}\}^*$ which each end at height 0 and which have no a or a^{-1} steps at height 0, such that $\overline{w_1 \dots w_n} = 1$. In this generating function, x counts the total length of $w_1 \dots w_n$ and y counts the total number of steps which end at height 0. Given such a loop l , let the baseloop l' be the subword consisting of all a and a^{-1} steps at t -height 0. Similarly to the previous algorithm, we let c_i be the number of steps in l' from a^{i-1} to a^i , and d_i be the number of steps in l' from a^{-i+1} to

a^{-i} . Then the length $|l'|$ of l' is equal to

$$\sum_{i=1}^m 2c_i + \sum_{j=1}^n 2d_j.$$

As in the previous subsection, for a given pair of sequences $c_1, \dots, c_m, d_1, \dots, d_n$, the number of such base loops is equal to

$$(7) \quad \binom{c_1 + d_1}{c_1} \prod_{i=1}^{m-1} \binom{c_i + c_{i+1} - 1}{c_i - 1} \prod_{j=1}^{n-1} \binom{d_j + d_{j+1} - 1}{d_j - 1}.$$

Let $l' = a_1 a_2 \dots a_{|l'|}$, where each $a_i \in \{a, a^{-1}\}$, and let $l = w_1 a_1 w_2 \dots a_{|l'|} w_{|l'|+1}$ be the decomposition where each step a_i is at t -height 0. We say that the a -height of one of the subwords w_i is equal to the integer p which satisfies $a^p = a_1 \dots a_i$. Then l is a loop if and only if for any height h , concatenating all of the words w_i at a -height h creates a loop. Note that each word w_i must have height 0 and have no a or a^{-1} steps at height 0. As in the previous section we define another generating function $\Lambda_d(x, y)$ by

$$\Lambda_d(x, y) = \sum H_{d_n} \prod_{j=1}^{n-1} x^{2d_{j+1}} y^{2d_{j+1}} \binom{d_j + d_{j+1} - 1}{d_j - 1} H_{d_j + d_{j+1}}(x),$$

where the sum is over all sequences n, d_1, d_2, \dots, d_n with $d_1 = d$. In the same way as in the previous section we get the following equations, which are essentially the same as (5) and (6).

$$(8) \quad \begin{aligned} G(x, y) &= \sum_{c,d=1}^{\infty} x^{2c+2d} y^{2c+2d} \binom{c+d}{c} H_{c+d+1}(x, y) \Lambda_d(x, y) \Lambda_c(x, y) \\ &\quad + 2 \sum_{d=1}^{\infty} x^{2d} y^{2d} H_d(x, y) \Lambda_d(x, y) \\ &\quad + H_1(x, y). \end{aligned}$$

$$(9) \quad \Lambda_d(x) = H_d(x, y) + \sum_{b=1}^{\infty} x^{2b} y^{2b} \binom{b+d-1}{d-1} H_{b+d}(x, y) \Lambda_b(x).$$

So now to calculate $G(x, y)$, we just need to calculate the generating functions $H_n(x, y)$. For each $k \in \mathbb{Z}_{\geq 0}$, let $J_k(x)$ be the generating function for loops in B which have exactly k steps which end at t -height 0, none of which are a or a^{-1} steps, and which never go below height 0. For each such word w , the number of ways of breaking it into n words w_1, w_2, \dots, w_n where each ends at height 0, such that $w_1 \dots w_n = w$ is equal to

$$\binom{k+n-1}{n-1}.$$

Therefore, we can calculate each generating function $H_n(x, y)$ in terms of the generating functions $J_k(x)$ as follows:

$$(10) \quad H_n(x, y) = \sum_{k=0}^{\infty} y^k \binom{k+n-1}{n-1} J_k(x).$$

Finally, we will calculate the generating functions $J_k(x)$. Trivially we have $J_0(x) = 1$. For $k > 0$, let l be a loop counted by $J_k(x)$. Then l must contain exactly k steps which end at height 0, which are not a or a^{-1} steps. Hence they must all be t^{-1} steps. Therefore, l decomposes as

$$l = tu_1t^{-1}tu_2t^{-1} \dots tu_kt^{-1},$$

where each word u_k ends at height 0 and never goes below height 0. Moreover, since l is a loop, we must have $\overline{u_1 \dots u_k} = 1$. Hence the word $u = u_1 \dots u_k$ is counted by the generating function $G(x, y)$. Moreover, if u contains m steps which end at height 0, then there are exactly

$$\binom{m+k-1}{k-1}$$

ways to decompose u into subwords u_1, \dots, u_k which each end at height 0. Hence we get the equation

$$(11) \quad J_k(x) = \sum_{m=0}^{\infty} x^{2k} \binom{m+k-1}{k-1} [y^m] G(x, y).$$

Now using equations (8), (9), (10) and (11) as well as the base case $J_0(x) = 1$, we can calculate the coefficients of $G(x, y)$ in polynomial time using a dynamic program. Finally we need to relate these coefficients to the total number of loops in B . We claim that for each n , the number of loops b_n of length n in B over the generating set $\{a, t, a^{-1}, t^{-1}\}$ is equal to

$$b_n = \sum_{m=0}^{\infty} \frac{n}{m} [y^m][x^n] G(x, y).$$

The reason for this is that the contribution to both sides of the equation from any set of n loops which are cyclic permutations of each other is the same. That is, if we take n loops $x_i \dots x_n x_1 \dots x_{i-1}$ for $1 \leq i \leq n$, and m of these are counted by $G(x, y)$, then they will each contribute $x^n y^m$ to $G(x, y)$, so altogether these will contribute n to both sides of the equation. If two or more of these loops are identical, then we must have $x_1 \dots x_n = (x_1 \dots x_p)^q$ for some p, q satisfying $pq = n$. In this case, assuming that q is maximal, the contribution to each side is n/q instead of n , since we overcounted by a factor of q .

Using the last equation we can quickly calculate the coefficients of the cogrowth generating function $C_B(x)$ using those of $G(x, y)$. In Table 1 we give the first 50 coefficients of this generating function. In fact we have 128 terms.

1
4
28
232
2092
19864
195352
1970896
20275692
211825600
2240855128
23952786400
258287602744
2806152315048
30686462795856
337490492639512
3730522624066540
41422293291178872
461802091590831904
5167329622166765872
58012358366319158872
653272479274904359312
7376993667962247094112
83518163933592420945440
947797532286760923097848
10779770914124700529470264
122856228305621394118000520
1402877847412263986004347872
16048147989560391552043686160
183892883412730524613883088808
2110556326150834244975990231512
24259510831181186885644198829344
279244563297679787781517160899820
3218641495385722409923501191862264
37146337262307758446419466115479416
429227600058421313330040967935014416
4965493663308539362541734301378311648
57506535582014868288482236767840209688
666700108804771886996957763509359246064
7737176908622194648339548498436658811432
89878279784970230837678375953110478795352
1045033044367535197025078407316665177933928
12161645115366917947524997117208173413019632
141653302005285175865456465524239660635389712
1651274058730064356309776255817393993665780288
19264448513399180870635082273788105896265150480
224919270246185854430934219198103161122414157760
2627954546552385827255336138747466100454012242528
30726935577139566309665785537931570627782996384120
359517978960007312327796870699755173605904761839752

TABLE 1. The first 50 coefficients of the cogrowth series for the Navas-Brin group B .

2.3. A General Algorithm. Before we describe the algorithm which we use for Thompson's group F , we will describe a general algorithm which can be applied to any group admitting certain functions which can be computed very quickly. In the next subsection we will describe how we apply this algorithm to F . This algorithm could also be applied to any of the other groups which we have discussed, however it would be much less efficient than the specific algorithms described previously in this section.

Our algorithm can be seen as a significantly more memory efficient version of the algorithm in [6]. First we describe that algorithm. Given a loop $\gamma = a_0 a_1 \dots a_{2n}$, where each $a_i \in V(\Gamma)$ and $a_{2n} = a_0 = e$, we define the midpoint of γ to be the vertex a_n . Then γ is made up of a walk of length n from e to its midpoint followed by a walk of length n from its midpoint to e . Hence, the number of loops in Γ of length $2n$ with midpoint m is the square of the number of walks of length n from e to m .

Using a simple dynamic program, the algorithm calculates the number of walks to each vertex in $B(e, n)$, the ball of radius n in Γ . Then one sums the squares of these numbers to calculate the number of loops of length $2n$. Note also that for each walk from e to m , there is a corresponding walk from e to m^{-1} , so it is only necessary to calculate the number of walks to either m or m^{-1} . The problem with this algorithm is that it is necessary to store a large proportion of the ball of radius n in memory at the same time. As a result it is essentially impossible to get any more than 24 coefficients of the cogrowth series for Thompson's group F using this algorithm. Our algorithm is very similar except that we only store the ball of radius k in memory, where $k \approx n/2$. Importantly, we do this without significantly increasing the running time of the program.

Let G be a group with inverse closed generating set S . Let $\Gamma(G, S)$ denote the Cayley graph of G with respect to the generating set S . We will often refer to this as simply Γ . We will assume that every loop has even length, however this algorithm could easily be altered to apply when this is not the case.

Let O be an object in the program which represents an element of G . We require the following functions to be implemented:

- $init()$. This returns an object O which represents the identity in G .
- $val(O)$. This returns a value which is uniquely determined by the element of G which the object O represents. In other words, $val(O_1) = val(O_2)$ if and only if O_1 and O_2 represent the same element of G .
- For each generator $\lambda \in S$, we have an operation $O.do_\lambda$. If O initially represents the element $g \in G$, this changes O to an object which represents $g\lambda$.
- For each generator $\lambda \in S$, we have a function $l_\lambda(O)$, defined by $l_\lambda(O) = |g\lambda| - |g|$, where g is the element of G which O represents. That is, $l_\lambda(O) = 1$ if applying λ moves g away from the identity.

The speed of our algorithm depends entirely on the efficiency of these functions. For Thompson's group our implementations of these all take constant time. Importantly, we do not require an inverse of val to be implemented.

Given these functions, the algorithm proceeds as follows:

Step 1: Assign an arbitrary order to the generating set S and set $k = \lceil \frac{n}{2} \rceil$.

Step 2: Using a simple dynamic program, construct an associative array A_{n-k} , implemented as a hash table, with a key value pair (k_g, a_g) for each element $g \in G$ within the ball of radius $n - k$. The key k_g is given by $val(O)$ where O is any object which represents g and the value a_g is equal to the number of walks of length $n - k$ in Γ from e to g . We will write $a_g = A[k_g]$. For a number x which is not a key in A_{n-k} , we set $A[x] = 0$.

Step 3: Construct a tree T_k which contains one vertex v_g for each element g of G within the ball of radius k , such that each vertex v_g , apart from v_e , is connected to exactly one vertex v_h satisfying $|h| = |g| - 1$, and $g = h\lambda$ for some $\lambda \in S$. If there are multiple possible choices of h , we choose the element h which minimises λ , according to the order we assigned in step 1. The edge (h, g) is then labelled with λ . Each vertex v_g is also labelled with the number $p(v_g)$ of paths of length k in Γ from e to g .

Step 4: We now create a function $numpaths(O, d)$ whose input is an object O and a positive integer d , which, assuming that $d = |g|$, outputs the number of paths of length n in Γ from e to g , where g is the group element represented by O . During the calculation of $numpaths$ the object O may change, but at the end it must represent the same group element g . Each path of length n from e to g^{-1} in Γ can be written in a unique way as a path of length k from e to some vertex h in Γ followed by a path of length $n - k$ from h to g^{-1} . For a given h , the number of these paths is equal to $p(v_h)A[k_{h^{-1}g^{-1}}] = p(v_h)A[k_{gh}]$. Hence, the number which we need to return is

$$\sum_{h \in G} p(v_h)A[k_{gh}].$$

Note also that the summand is 0 unless $|h| \leq k$ and $|gh| \leq n - k$, so we only need to sum over values of h which satisfy these two inequalities. To do this we perform a depth first search of the tree T_k , skipping any sections where we can be sure that there are no vertices v_h such that h satisfies the two inequalities. We start the search at the root vertex v_e of T_k and initialise $r = 0$ and $total = 0$. Whenever we move from a vertex v_h to $v_{h\lambda}$ we change d to $d + l_\lambda(O)$ and then apply the operation $O.do_\lambda$. That way whenever we are at a vertex v_h , the object O represents gh and $d = |gh|$. We also increase x by 1 whenever we move to a child vertex and decrease x by 1 when we backtrack so that we always have $x = |h|$. Then we add $p(v_h)A[k_{gh}] = p(v_h)A[val(O)]$ to the sum $total$ if and only if $d \leq n - k$, since $x = |h| \leq k$ for every vertex v_h in T_k . Since d decreases by at most 1 when we move to a child vertex, and x always increases by 1, the value $x + d$ never decreases when we move to a child vertex. So if $x + d > n$ when we are at a vertex v_h , then we do not traverse the children of v_h . At the end of the search we return to the root vertex so that O is back to its original value and then return the value $total$.

Step 5: For the last step we just need to add up the value of $numpaths$ for every vertex g in the ball of radius n such that $|g|$ has the same parity as n . To accomplish this we perform a depth first search of the tree T_n , which is defined in the same way as T_k . However, we do not explicitly construct T_n as doing so would use too much memory. In order to perform the depth first search, we just need a function $isedge_\lambda(O)$ for each $\lambda \in S$ which returns 1 if and only if there is an outward edge from v_g to $v_{g\lambda}$ in T_n , where g is the group element that O represents. This will be the case if and only if $|g\lambda| = |g| + 1$

and $|g\lambda\mu| = |g\lambda| + 1$ for each $\mu \in S$ with $\mu < \lambda^{-1}$. We test this using the functions l_λ , do_λ and l_μ . During the depth first search, we keep track of the distance $d = |g|$, where g is the group element represented by O . Now, to calculate the number $numloops$ of loops of length $2n$, we first set $numloops = 0$, then run the depth first search, and when we visit each vertex of T_n , add $numpaths(O, d)^2$ to $numloops$. At the end of this process $numloops$ is equal to the number of loops of length $2n$, so we return $numloops$ and terminate the algorithm.

The advantage of this algorithm is that it only stores T_k and A_{n-k} in memory, rather than all of T_n . This also allows us to parallelise step 5.

2.4. Thompson's group F . In this section we describe how the object O , the operation do_λ and the functions val and l_λ are implemented for Thompson's group F . We use the standard generating set $S = \{a, b, a^{-1}, b^{-1}\}$, which yields the presentation

$$F = \langle a, b \mid a^2ba^{-2} = baba^{-1}b^{-1}, a^3ba^{-3} = ba^2ba^{-2}b^{-1} \rangle.$$

For O we use the forest representation given by Belk and Brown in [2]. We simultaneously store the forest diagram as a graph P as well as a pair of binary strings a, b . A forest diagram is defined as a pair of sequences of binary trees, with one tree highlighted in each sequence. A single binary tree with m leaves corresponds to a unique binary string s of length $2m - 2$ with the property that s has an equal number of 1's and 0's and the number of 1's in any initial substring is at least equal to the number of 0's in that substring. This is defined by doing a depth first search of the tree and writing a 1 whenever we move down an edge from a vertex to its left subtree and writing a 0 whenever we backtrack along such an edge. Now to convert a sequence of binary trees to a binary string, we first convert each individual tree to a binary string, insert the string 01 before each such string, then concatenate the results. We then change the 01 before the string corresponding to the highlighted tree to 00. This is how the strings a and b are defined. We also store the numbers p_a and p_b in O , which define the positions of the 00 before the highlighted tree in each of a and b . The strings a and b each have length at most $2n$, so they can be represented as 64 bit numbers as long as $n \leq 32$. The operation do_λ is defined easily for the effect on the graph P . The effect on the binary strings a and b is a bit more complicated and requires some bit shifting. The entire length of an element of Thompson's group F can be determined by its forest diagram, as shown in [2], so we could use this to determine l_λ by using the graph P and simply subtracting the calculated length $|g|$ from the length we calculate for $|g\lambda|$. In fact we do it more efficiently than this, as the difference $|g\lambda| - |g|$ is determined entirely by the highlighted tree and the surrounding trees. Finally, $val(O)$ simply returns the pair (a, b) .

In Table 2 we give the first 32 coefficients of the cogrowth generating function for Thompson's group F . This is 7 further terms than given in [18].

Coefficients
1
4
28
232
2092
19884
196096
1988452
20612364
217561120
2331456068
25311956784
277937245744
3082543843552
34493827011868
389093033592912
4420986174041164
50566377945667804
581894842848487960
6733830314028209908
78331435477025276852
915607264080561034564
10750847942401254987096
126768974481834814357308
1500753741925909645997904
17833339046478612301547884
212663448005862463186139032
2544535423071442709522261116
30542557512715560857221200908
367718694478039302564802454628
4439941127401928226610731571976
53756708216952135677787623701460

TABLE 2. Terms in the cogrowth sequence of Thompson's group F .

3. POSSIBLE COGROWTH OF THOMPSON'S GROUP

In this section we will show that if a_0, a_1, \dots is the cogrowth sequence for Thompson's group F , then for any real numbers $a < 1$ and $\lambda > 1$, the inequality

$$a_n < 16^n \lambda^{-n^a}$$

holds for all sufficiently large integers n . As a result, if Thompson's group is amenable, then the sequence cannot grow at the rate

$$16^n \lambda^{-n^a},$$

For any fixed $a < 1$. This result follows quite readily from results in [21] and [20], however we will need some definitions before we can see how they apply. Let G be a group with finite generating set S . Then we define the function $\phi_S : \mathbb{Z}_{>0} \rightarrow \mathbb{R}_{>0}$ by setting $\phi_S(n)$ to be the probability that a random walk in (G, S) of length $2n$ finishes at the origin. In other words, $|S|^{2n} \phi_S(n)$ is the number of loops of length $2n$ in the Cayley graph $\Gamma(G, S)$.

Now, for two different (non-increasing) functions ϕ_1 and ϕ_2 , we say that $\phi_1 \preceq \phi_2$, if there is some $C \in \mathbb{R}_{>0}$ such that $\phi_1(n) \leq C \phi_2(n/C)$, where each ϕ_i is extended to the reals by linear interpolation. Finally we say that $\phi_1 \approx \phi_2$ if both $\phi_1 \preceq \phi_2$ and $\phi_2 \preceq \phi_1$. We recall Theorem 3.1 from [20]:

Theorem 3.1. *Let G be a group with finite, symmetric generating set S and let H be a subgroup of G and let T be a finite symmetric generating set of H . Then*

$$\phi_S \preceq \phi_T.$$

The other result we need concerns wreath products with \mathbb{Z} . In [21], Pittet and Saloff-Coste show (in a remark just below Theorem 8.11) that for a finite generating set T of $\mathbb{Z} \wr_d \mathbb{Z}$, we have

$$\phi_T(n) \approx \exp\left(-n^{d/(d+2)}(\log n)^{2/(d+2)}\right).$$

Now, since $\mathbb{Z} \wr_d \mathbb{Z}$ is a subgroup of Thompson's group F , we must have

$$\phi_S(n) \preceq \phi_T(n) \approx \exp\left(-n^{d/(d+2)}(\log n)^{2/(d+2)}\right),$$

where S is the standard generating set of F . Hence, for any positive integer d , there is a positive real number C such that

$$\phi_S(n) \leq C \exp\left(-(n/C)^{d/(d+2)}(\log(n/C))^{2/(d+2)}\right).$$

Now we are ready to prove our theorem.

Theorem 3.2. *Let a_n be the number of loops of length $2n$ in the standard Cayley graph for Thompson's group. Then for any real numbers $a < 1$ and $\lambda > 1$, the inequality*

$$a_n < 16^n \lambda^{-n^a}$$

holds for all sufficiently large integers n .

Proof. Let d be a positive integer such that $\frac{d}{d+2} > a$. Then there is some $C \in \mathbb{R}_{>0}$ such that

$$\phi_S(n) \leq C \exp\left(-(n/C)^{d/(d+2)} \log(n/C)^{2/(d+2)}\right)$$

for all $n \in \mathbb{Z}_{>0}$. For n sufficiently large, we have $\log(n/C) > 0$, so

$$\begin{aligned} C \exp\left(-\left(\frac{n}{C}\right)^{d/(d+2)} \log\left(\frac{n}{C}\right)^{2/(d+2)}\right) &< C \exp\left(-\left(\frac{n}{C}\right)^{d/(d+2)}\right) \\ &= \exp\left(\log(C) - C^{-d/(d+2)} n^{d/(d+2)}\right), \end{aligned}$$

Hence, for all n sufficiently large we have

$$\phi_S(n) < \exp(-n^\alpha).$$

Therefore,

$$a_n = 16^n \phi_S(n) < 16^n \exp(-n^\alpha)$$

□

Note that the same result holds if we replace Thompson's group F with the Navas-Brin group B , since it also contains every wreath product $\mathbb{Z} \wr_d \mathbb{Z}$ as a subgroup.

4. MOMENTS

In [18], Haagerup, Haagerup and Ramirez-Solano prove that the cogrowth sequence a_0, a_1, \dots for Thomson's group F is the sequence of moments of some probability measure μ on $[0, \infty)$, in other words, the sequence is a Stieltjes moment sequence. In fact, their proof applies to the cogrowth series of any (locally finite) Cayley graph Γ . In this section, we generalise the result further, to any locally finite graph. First we give some background on the Stieltjes and Hamburger moment problems.

4.1. Stieltjes and Hamburger moment sequences. In the following, for the sequence $\mathbf{a} = a_0, a_1, \dots$, and $n \geq 0$, we define the matrix $H_\infty^{(n)}(\mathbf{a})$ by

$$H_\infty^{(n)}(\mathbf{a}) = \begin{bmatrix} a_n & a_{n+1} & a_{n+2} & \cdots \\ a_{n+1} & a_{n+2} & a_{n+3} & \cdots \\ a_{n+2} & a_{n+3} & a_{n+4} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Theorem 4.1. (*Stieltjes [23], GantmakherKrein [9]*) *For a sequence $\mathbf{a} = a_0, a_1, \dots$, the following are equivalent:*

- *There exists a positive measure μ on $[0, \infty)$ such that*

$$a_n = \int x^n d\mu(x).$$

- *The matrices $H_\infty^{(0)}(\mathbf{a})$ and $H_\infty^{(1)}(\mathbf{a})$ are both positive semidefinite.*
- *There exists a sequence of real numbers $\alpha_0, \alpha_1, \dots \geq 0$ such that the generating function $A(t)$ for the sequence a_0, a_1, \dots satisfies*

$$A(t) = \sum_{n=0}^{\infty} a_n t^n = \frac{\alpha_0}{1 - \frac{\alpha_1 t}{1 - \frac{\alpha_2 t}{1 - \dots}}}$$

A sequence which satisfies the conditions of the theorem above is called a Stieltjes moment sequence.

Theorem 4.2. *For a sequence $\mathbf{a} = a_0, a_1, \dots$, the following are equivalent:*

- *There exists a positive measure μ on $(-\infty, \infty)$ such that*

$$a_n = \int x^n d\mu(x).$$

- *The matrix $H_\infty^{(0)}(\mathbf{a})$ is positive semidefinite.*

A sequence which satisfies the conditions of the theorem above is called a Hamburger moment sequence. From either definition of Hamburger moment sequence, it follows immediately that any Stieltjes moment sequence is a Hamburger moment sequence. Carleman's condition states that the measure μ is unique if

$$\sum_{n=0}^{\infty} a_{2n}^{-\frac{1}{2n}} = +\infty,$$

This is certainly true when the sequence grows at most exponentially, as is the case for all of our examples. For Stieltjes moment sequences, the following weaker condition implies that the measure μ is unique:

$$\sum_{n=0}^{\infty} a_n^{-\frac{1}{2n}} = +\infty.$$

For a Hamburger moment sequence \mathbf{a} , which grows at most exponentially, the radius of convergence of $A(t) = a_0 + a_1t + a_2t^2 + \dots$ is equal to

$$\frac{1}{|\mu|}.$$

In particular, this means that if \mathbf{a} is a Stieltjes moment sequence, the exponential growth rate of the sequence is equal to the minimum value in the support of μ .

One benefit of proving that a sequence \mathbf{a} is a Stieltjes moment sequence is that it allows us to compute good lower bounds for the exponential growth rate of the sequence using only finitely many terms. This method was described in [18], but we repeat the description here, using the continued fraction form of \mathbf{a} . We consider the generating function

$$A(t) = \sum_{n=0}^{\infty} a_n t^n = \frac{\alpha_0}{1 - \frac{\alpha_1 t}{1 - \frac{\alpha_2 t}{1 - \dots}}}.$$

Using the terms a_0, \dots, a_n , we calculate the terms $\alpha_0, \dots, \alpha_n$. It is easy to see that $A(t)$ is nondecreasing in each α_j . Hence, the minimum possible value $A_n(t)$ is achieved by setting $\alpha_{n+1}, \alpha_{n+2}, \dots$ to 0. Therefore, the radius of convergence t_c of $A(t)$ is bounded above by the radius of convergence t_n of $A_n(t)$. Therefore, $b_n = 1/t_n$ is a lower bound for the exponential growth rate of \mathbf{a} . It is easy to check that the sequence b_1, b_2, \dots is

nondecreasing, and $b_n^n > a_n/a_0$. It follows that this sequence of lower bounds converges to the exponential growth rate of \mathbf{a} .

If we assume further that the sequences $\alpha_0, \alpha_2, \alpha_4, \dots$ and $\alpha_1, \alpha_3, \dots$ are non-decreasing, as seems to be true for many of the cases we consider, we can get stronger lower bounds for the growth rate by setting $\alpha_{n+1}, \alpha_{n+3}, \dots$ to α_{n-1} and $\alpha_{n+2}, \alpha_{n+4}, \dots$ to α_n . For this sequence the exponential growth rate of corresponding sequence \mathbf{a} is $(\sqrt{\alpha_n} + \sqrt{\alpha_{n-1}})^2$.

4.2. Applications of moments to the cogrowth series. Here we describe how to compute lower bounds for the growth rate of the cogrowth sequence of Thompson's group F .

Theorem 4.3. *Let Γ be a locally finite graph with a fixed base vertex v_0 . For each $n \in \mathbb{Z}_{\geq 0}$, let t_n be the number of loops of length n in Γ which start and end at v_0 . Then there exists a probability measure μ on \mathbb{R} such that for each $n \in \mathbb{Z}_{\geq 0}$, the n th moment of μ is given by*

$$\int_{-\infty}^{\infty} x^n d\mu = t_n.$$

In other words, t_0, t_1, \dots is a Hamburger moment sequence.

Proof. The sequence $\mathbf{t} = t_0, t_1, \dots$ is a Hamburger moment sequence if and only if the matrix

$$H_{\infty}^{(0)}(\mathbf{t}) = \begin{bmatrix} t_0 & t_1 & t_2 & \dots \\ t_1 & t_2 & t_3 & \dots \\ t_2 & t_3 & t_4 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

is positive semidefinite. To prove this, we will show that this is the matrix representation of a positive definite bilinear form.

Let M be the inner product space over \mathbb{R} defined by the orthonormal basis $\{b_v | v \in V(\Gamma)\}$. For each $n \in \mathbb{Z}$ we let $x_n \in M$ be the element defined by

$$x_n = \sum_{v \in V(\Gamma)} p_v b_v,$$

where p_v is the number of paths of length n in Γ from v_0 to v . Then it is easy to see that for any non-negative integers m and n , the value $\langle x_n, x_m \rangle$ is equal to the number t_{m+n} of paths of length $m+n$ in Γ from v_0 to itself. Now let X be the subspace of M spanned by $\{x_0, x_1, \dots\}$. Then A is the matrix representation of the inner product $\langle \cdot, \cdot \rangle$, restricted to X , with respect to the spanning set $\{x_0, x_1, \dots\}$. Therefore, $H_{\infty}^{(0)}(\mathbf{t})$ is positive semidefinite. Note that if $\{x_0, x_1, \dots\}$ are linearly independent, then $H_{\infty}^{(0)}(\mathbf{t})$ is positive definite. \square

Theorem 4.4. *Let $C \in \mathbb{Z}_{>0}$ and let Γ be a graph with a fixed base vertex v_0 , such that each vertex in Γ has degree at most C . For each $n \in \mathbb{Z}_{\geq 0}$, let t_n be the number of loops of length n in Γ which start and end at v_0 . There exists a probability measure μ on $\mathbb{R}_{>0}$ such that for each $n \in \mathbb{Z}_{\geq 0}$, the n th moment of μ is equal to t_{2n} . In other words, t_0, t_2, t_4, \dots is a Stieltjes moment sequence.*

Moreover, μ is unique and its support is contained in the interval $[0, C^2]$.

Proof. In order to show that the sequence $\mathbf{s} = t_0, t_2, t_4, \dots$ is a Stieltjes moment sequence, it suffices to prove that the two matrices

$$H_\infty^{(0)}(\mathbf{s}) = \begin{bmatrix} t_0 & t_2 & \dots \\ t_2 & t_4 & \dots \\ \vdots & \vdots & \ddots \end{bmatrix} \quad \text{and} \quad H_\infty^{(1)}(\mathbf{s}) = \begin{bmatrix} t_2 & t_4 & \dots \\ t_4 & t_6 & \dots \\ \vdots & \vdots & \ddots \end{bmatrix}$$

are positive semidefinite. From the previous theorem, we know that the matrix

$$H_\infty^{(0)}(\mathbf{t}) = \begin{bmatrix} t_0 & t_1 & t_2 & \dots \\ t_1 & t_2 & t_3 & \dots \\ t_2 & t_3 & t_4 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

is positive semidefinite. Hence any principal submatrix of $H_\infty^{(0)}(\mathbf{t})$ (using the same rows and columns) is also positive semidefinite. Since both the matrices $H_\infty^{(0)}(\mathbf{s})$ and $H_\infty^{(1)}(\mathbf{s})$ are such principal submatrices of $H_\infty^{(0)}(\mathbf{t})$, each of these matrices is positive semidefinite. Therefore, the sequence t_0, t_2, t_4, \dots is a Stieltjes moment sequence. Now, since each vertex of the graph has degree at most C , the number of paths of length n is at most C^n . Hence we have the inequality

$$\int_0^\infty x^n d\mu = t_{2n} \leq C^{2n}.$$

Therefore, the support of μ must be contained in the interval $[0, C^2]$. This also implies that μ is unique. \square

In particular, if we let G be a finitely generated group, with inverse closed generating set S , and Γ be the corresponding Cayley graph, then the even terms of the cogrowth sequence for Γ form a Stieltjes moment sequence. Moreover, each vertex has degree $|S|$ so the support of the corresponding measure μ is contained in the interval $[0, |S|]$. As described in the previous subsection, we can compute lower bounds b_n for the exponential growth rate of any such sequence. Turning our attention to Thompson's group, using 31 terms of the cogrowth sequence, we have computed the corresponding terms $\alpha_0, \alpha_1, \dots, \alpha_{31}$. Using these we have computed the rigorous lower bound $b_{31} \approx 13.269$ for the exponential growth rate of the cogrowth sequence of Thompson's group. If we assume that the sequences $\alpha_0, \alpha_2, \dots$ and $\alpha_1, \alpha_3, \dots$ are increasing, we get the stronger lower bound $(\sqrt{\alpha_{30}} + \sqrt{\alpha_{31}})^2 \approx 13.706$. In Section 8 below we extrapolate the sequence of bounds $\{b_n\}$ to estimate the growth constant μ , and find $\mu \approx 15.0$.

5. SERIES ANALYSIS

We have series for six groups, which we will consider in order. Firstly, the group \mathbb{Z}^2 , then the Heisenberg group, the lamplighter group $L = \mathbb{Z}_2 \wr \mathbb{Z}$, the two groups $\mathbb{Z} \wr \mathbb{Z}$ and $(\mathbb{Z} \wr \mathbb{Z}) \wr \mathbb{Z}$, the Navas-Brin group B [19, 4] and finally Thompson's group F . We will analyse each of these in turn.

In all cases our initial analysis is based on the behaviour of the ratio of successive terms, with other methods deployed as appropriate. In the simplest situation we consider, which is when the asymptotic form of the coefficients is $c_n \sim c \cdot \mu^n \cdot n^g$, one has that the *ratio* of successive coefficients is asymptotically linear when plotted against $1/n$, as

$$(12) \quad r_n = \frac{c_n}{c_{n-1}} = \mu \left(1 + \frac{g}{n} + o\left(\frac{1}{n}\right) \right).$$

It is therefore natural to plot the ratios r_n against $1/n$. If the correction term $o\left(\frac{1}{n}\right)$ can be ignored³, such a plot will be linear, with gradient $\mu \cdot g$, and intercept μ at $1/n = 0$. If the growth constant μ is known, or can be guessed, better estimates of the exponent g can be made by extrapolating the sequence

$$g_n = (r_n/\mu - 1) \cdot n = g + o(1).$$

More complicated asymptotic forms for the coefficients can give rise to different expressions for the ratios, as we show below.

5.1. The group \mathbb{Z}^2 . For the group \mathbb{Z}^2 , the coefficients of the cogrowth series are known exactly, $c_n = \binom{2n}{n}^2$, and so the ratio of successive terms is

$$r_n = \frac{c_n}{c_{n-1}} = 16 \left(1 - \frac{1}{n} + \frac{1}{4n^2} \right).$$

A plot of the ratios against $1/n$ is shown in Figure 1, based on the first 50 coefficients. It is clearly going to the expected limit of 16. The exponent g should be -1 , and we plot estimators g_n against $1/n$ in Figure 2, which is also clearly going to the expected limit -1 . This corresponds to a logarithmic singularity of the generating function,

$$C_{\mathbb{Z}^2}(x) \sim c \cdot \log(1 - 16x).$$

For this simple example one can do much better by using the package *gfun*, available in Maple, and asking for the underlying ordinary differential equation for the generating function, given the first 20 or so coefficients. In this way one immediately obtains the result for the generating function

$$C_{\mathbb{Z}^2}(x) = \sum c_n x^n = 2\mathbf{K} \left(\frac{4\sqrt{x}}{\pi} \right),$$

where \mathbf{K} is the complete elliptic integral of the first kind.

5.2. The Heisenberg group. We have calculated 90 terms of the generating function, and show that this is sufficient to obtain a very precise asymptotic representation of the coefficients. The leading order asymptotics of the coefficients is known [10] to be $c_n \sim 16^n/(2n^2)$, corresponding to a generating function

$$C_{\text{Heisenberg}} \sim \frac{1}{2}(1 - 16x) \log(1 - 16x).$$

We have analysed this series in the same way as described above for the group \mathbb{Z}^2 .

³In the simplest cases, such as the present one, the correction term will be $O\left(\frac{1}{n^2}\right)$.

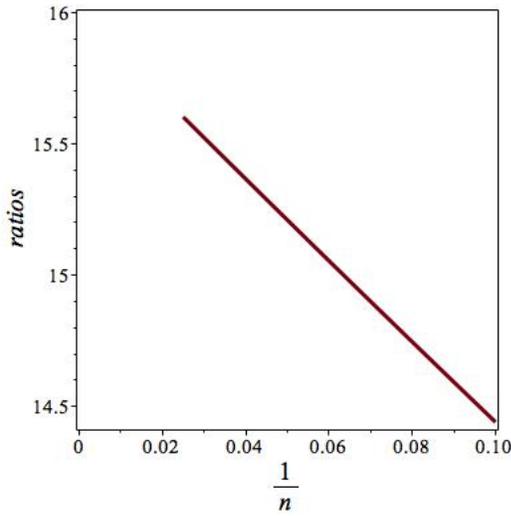


FIGURE 1. Plot of \mathbb{Z}^2 ratios against $1/n$.

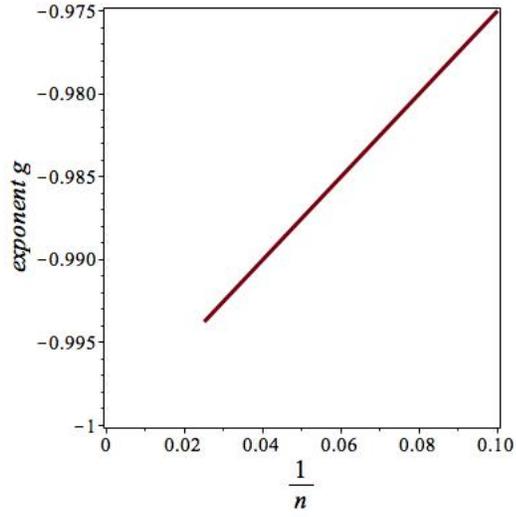


FIGURE 2. Estimators of exponent g for \mathbb{Z}^2 vs. $1/n$.

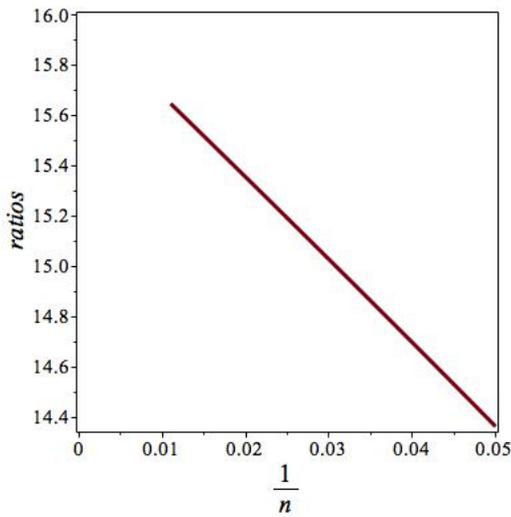


FIGURE 3. Plot of Heisenberg group ratios against $1/n$.

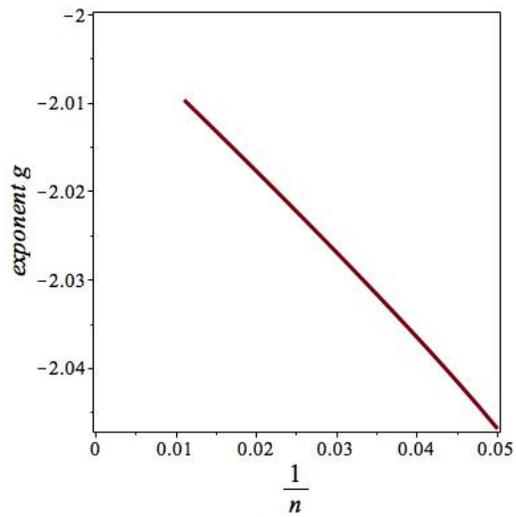


FIGURE 4. Estimators of exponent g for the Heisenberg group vs. $1/n$.

A plot of the ratios against $1/n$ is shown in Figure 3. It is clearly going to the expected limit of 16. The exponent g should be -2 , and we plot estimators g_n against $1/n$ in Figure 4, which are also clearly going to the expected limit -2 .

In order to obtain higher-order asymptotic terms, we subtract the known leading-order term from the coefficients, forming the sequence

$$c^{(1)}(n) = c_n - 16^n/(2n^2).$$

A ratio analysis of this sequence strongly suggests that $c^{(1)}(n) \sim \text{const}/n$, implying that $c_n \sim 16^n/(2n^2) + \text{const.}/n^3$. Such behaviour is consistent with a simple algebraic singularity of the generating function. Accordingly, we attempted a linear fit to the assumed form $c_n/16^n = 1/(2n^2) + k_1/n^3 + k_2/n^4 + k_3/n^5$. We did this by solving the linear system given by setting $n = m - 1$, $n = m$, $n = m + 1$ in the preceding equation, and solving for k_1 , k_2 , k_3 , with m ranging from 20 to the maximum possible value 89. We obtain an m -dependent sequence of estimates of the amplitudes k_1 , k_2 , k_3 , which we extrapolated against appropriate powers of $1/m$.

In this way we estimate $k_1 = 0.93341$, $k_2 = 1.530$, and $k_3 = 3.30$, where we expect errors in these estimates to be confined to the last quoted digit.

To summarise, we find the asymptotics of the coefficients of the cogrowth series of the Heisenberg group to be

$$c_n = 16^n \left(\frac{1}{2n^2} + \frac{0.93341}{n^3} + \frac{1.530}{n^4} + \frac{3.30}{n^5} + O\left(\frac{1}{n^6}\right) \right).$$

5.3. The lamplighter group. The lamplighter group L is the wreath product of the group of order two with the integers, $L = \mathbb{Z}_2 \wr \mathbb{Z}$. From [22] we know that for this group,

$$(13) \quad c_n \sim c \cdot 9^n \cdot \kappa^{n^{1/3}} \cdot n^{1/6}.$$

So in this example we see the presence of a stretched-exponential term, $\kappa^{n^{1/3}}$, which makes the analysis more difficult. As remarked above, we have generated 201 terms of the cogrowth series, and show how these terms can be used to estimate the asymptotic behaviour of the coefficients.

If the coefficients of a series include a stretched-exponential term, so that

$$a_n \sim c \cdot \mu^n \cdot \kappa^{n^\sigma} \cdot n^g,$$

with $0 < \sigma, \kappa < 1$, then the ratio of successive terms behaves as

$$r_n = \frac{a_n}{a_{n-1}} \sim \mu \left(1 + \frac{\sigma \log \kappa}{n^{1-\sigma}} + \frac{g}{n} + \dots \right).$$

Experimentally, the presence of such a stretched-exponential term is signalled by the fact that the ratio plots against $1/n$ exhibit curvature, and that this curvature can be eliminated, or at least substantially reduced, by plotting the ratios against $1/n^{1-\sigma}$, where σ is roughly estimated by choosing its value so as to maximise linearity. This theory is developed in greater detail, along with several examples, in [14].

Because of the presence of two terms in the ratio plots, one of order $O(n^{\sigma-1})$ the other of order $O(1/n)$, there is some competition between these two terms, which can make it

difficult to estimate the value of σ just from the linearity of the ratio plots. So we first eliminate the $O(1/n)$ term by calculating the modified ratios

$$(14) \quad r_n^{(1)} = n \cdot r_n - (n-1) \cdot r_{n-1} = \mu \left(1 + \frac{\sigma^2 \log \kappa}{n^{1-\sigma}} + o\left(\frac{1}{n}\right) \right).$$

In Figure 5 we show the modified ratios plotted against $1/n^{2/3}$, which is seen to be linear, and extrapolating to the known growth constant of 9. While not shown, we also plotted the modified ratios against $1/\sqrt{n}$ and against $1/n^{3/4}$. These were visibly convex upward and concave downward, respectively. One would conclude that $1/2 < \sigma < 3/4$, and bearing in mind that in all known such behaviour, σ is a simple rational fraction (arguably simply related to dimensionality), one would conjecture that $\kappa = 2/3$. However, we can also estimate the value of σ by other means.

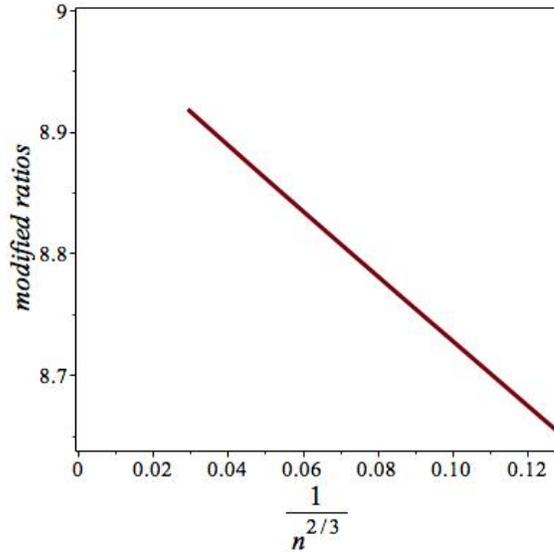
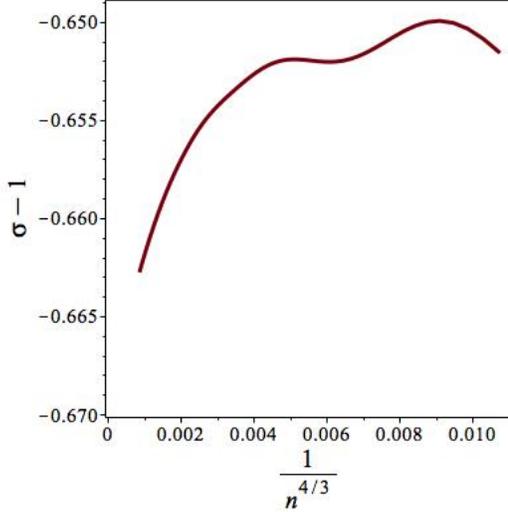
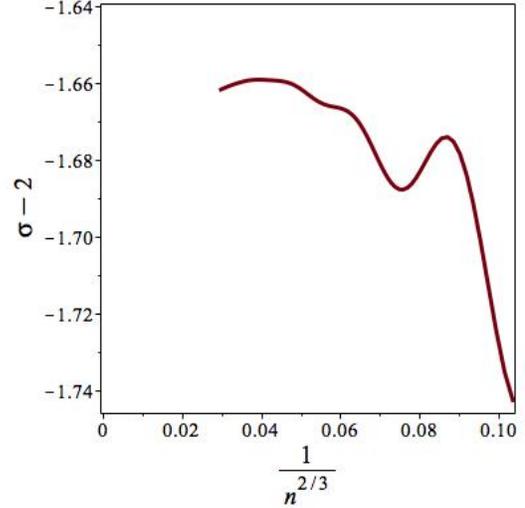


FIGURE 5. Modified lamplighter group ratios vs. $n^{-2/3}$.

If we assume $\mu = 9$, then from (14) it follows that a plot of $l_n = \log |1 - r_n^{(1)}/\mu|$ against $\log(n)$ should be linear with gradient $\sigma - 1$. This plot (not shown) is indeed visually linear. To calculate the gradient, which will vary slightly with n , we calculate the local gradient $(l_n - l_{n-1})/(\log(n) - \log(n-1))$, and show this plotted against $1/n^{4/3}$ in Figure 6. This plot is clearly going to a limit very close to $-2/3$, as expected.

One can also find estimators for the exponent σ without assuming or knowing the value of the growth constant μ . Taking the ratio of the modified ratios eliminates the growth constant μ , so that

$$r_n^{(2)} = \frac{r_n^{(1)}}{r_{n-1}^{(1)}} = 1 - \frac{\sigma^2(1-\sigma) \log \kappa}{n^{2-\sigma}} + o(n^{\sigma-2}).$$

FIGURE 6. Estimates of $\sigma - 1$ vs. $n^{-4/3}$.FIGURE 7. Estimates of $\sigma - 2$ vs. $n^{-2/3}$.

So a plot of $\log |r_n^{(2)} - 1|$ against $\log n$ should be linear with gradient $\sigma - 2$. As above, we don't show this uninteresting linear plot, but instead show the local gradient, plotted against $1/n^{2/3}$, in Figure 7, which appears to be going to a value around -1.67 , consistent with the known exact value $-5/3$.

Assuming the values $\mu = 9$, and $\sigma = 1/3$, we estimate the remaining parameters in the asymptotic expression by direct fitting to the logarithm of the coefficients. From $c_n \sim c \cdot 9^n \cdot \kappa^{n^{1/3}} \cdot n^g$ we get

$$\log c_n - n \cdot \log 9 \sim n^{1/3} \cdot \log \kappa + g \cdot \log n + \log c.$$

As in the preceding analysis of the Heisenberg group coefficients, we fit successive triples of coefficients to get estimates of the three unknowns, $\log \kappa$, g and $\log c$. The results are shown in Figures 8, 9, and 10 respectively.

From these plots, we estimate $\log \kappa \approx -2.78$, $g \approx 0.17$, and $\log c \approx -0.6$. If we use the fact that we know that the exponent $g = 1/6$, we can get refined estimates of the remaining parameters, giving $\log \kappa \approx -2.775$, and $\log c \approx -0.55$, so that $\kappa \approx 0.0623$, and $c \approx 0.58$. As far as we are aware, these two constants have not previously been estimated.

5.4. Analysis of group $\mathbb{Z} \wr \mathbb{Z}$. As discussed in the introduction, for the groups $\mathbb{Z} \wr_d \mathbb{Z}$, there is an additional logarithmic factor associated with the stretched-exponential term. For $d = 1$ the group $\mathbb{Z} \wr \mathbb{Z}$ has coefficients that behave as

$$a_n \sim \text{const} \cdot \mu^n \cdot \kappa^{n^\sigma \log^\delta n} \cdot n^g, \text{ with } \sigma = 1/3 \text{ and } \delta = 2/3.$$

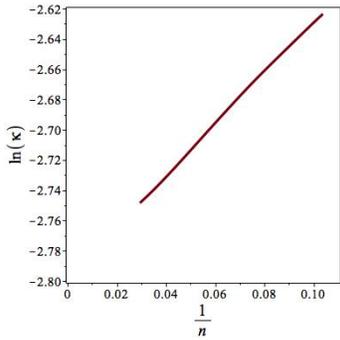


FIGURE 8. Estimates of $\log \kappa$ vs. $1/n$.

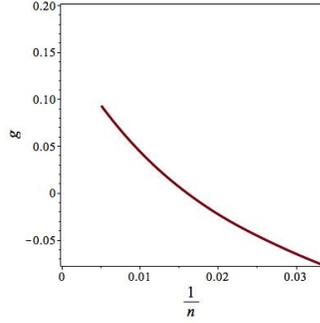


FIGURE 9. Estimates of exponent g vs. $1/n$. The exact value is $1/6$.

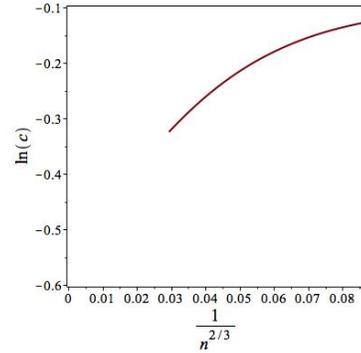


FIGURE 10. Estimates of $\log c$ vs. $n^{-2/3}$.

It follows that the ratio of successive coefficients behaves as

$$(15) \quad r_n = \frac{a_n}{a_{n-1}} \sim \mu \left(1 + \frac{\sigma \cdot \log \kappa \cdot \log^\delta n}{n^{1-\sigma}} + \frac{\delta \cdot \log \kappa \cdot \log^{\delta-1} n}{n^{1-\sigma}} + \frac{g}{n} + \dots \right).$$

We have generated series to order x^{276} for this group. A simple ratio plot against $1/n$ is strongly concave downwards. Plotting the ratios against $1/n^{2/3}$ gives a plot which is much closer to linear, but still displays a slight concavity. A simple ratio plot against $1/\sqrt{n}$ by contrast, displays slight convexity.

As we noted in our analysis of the lamplighter group, the term g/n in eqn. (15) also makes a contribution (as does the logarithmic term $\log^\delta n$), so a clearer picture emerges if this term is eliminated, which we do by forming the modified ratios (14), which behave in this case as

$$(16) \quad r_n^{(1)} = \mu \left(1 + \frac{\log \kappa}{9n^{2/3}} \left(\log^{2/3} n + 4 \log^{-1/3} n - 2 \log^{-4/3} n \right) + o(n^{-5/3+\epsilon}) \right).$$

Plots of the modified ratios are shown in Figures 11, 12, and 13, against $1/\sqrt{n}$, $1/n^{2/3}$ and $1/n^{3/4}$ respectively. It is clear that the plot against $1/n^{2/3}$ is the closest to linear, corresponding to $\kappa = 1/3$. However, there is still some downward concavity, due to the associated logarithmic terms. To see this even more clearly, we show in Figure 14 a plot of the modified ratios against $(\log^{2/3} n + 4 \log^{-1/3} n - 2 \log^{-4/3} n) / n^{2/3}$, which is the expected asymptotic behaviour, see (16). This is indistinguishable from linearity.

To date we haven't tried to estimate μ , known to be exactly 16. One way to do this is from the modified ratio plots shown above. All are seen to be tracking towards a value very close to 16.

It is also possible to estimate the exponent σ directly from the ratios, even without knowing the dominant exponential growth constant μ . One first forms the ratio of successive

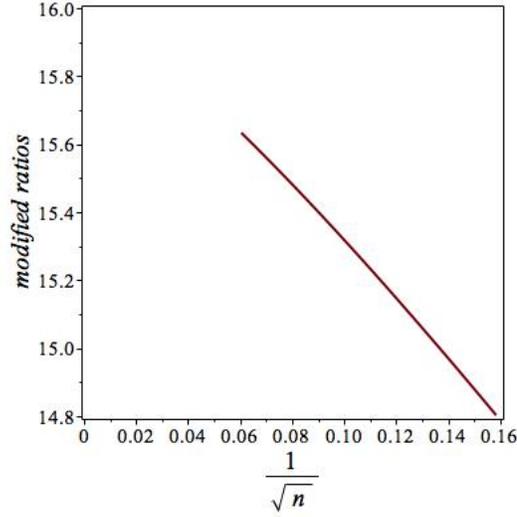


FIGURE 11. Modified ratios for $\mathbb{Z} \wr \mathbb{Z}$ vs. $1/\sqrt{n}$.

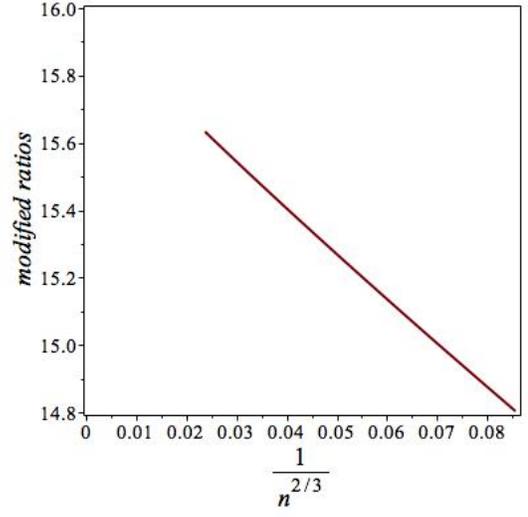


FIGURE 12. Modified ratios for $\mathbb{Z} \wr \mathbb{Z}$ vs. $n^{-2/3}$.

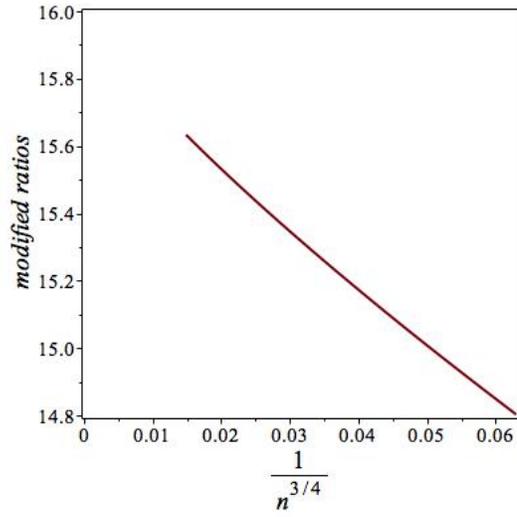


FIGURE 13. Modified ratios for $\mathbb{Z} \wr \mathbb{Z}$ vs. $n^{-3/4}$.

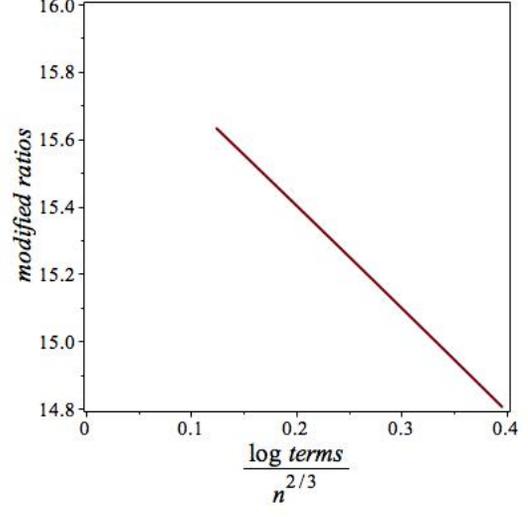


FIGURE 14. Modified ratios for $\mathbb{Z} \wr \mathbb{Z}$ vs. $(\log^{2/3} n + 4 \log^{-1/3} n - 2 \log^{-4/3} n) n^{-2/3}$.

ratios, so that

$$(17) \quad rr_n^{(1)} = \frac{r_n}{r_{n-1}} = 1 + \frac{\log \kappa \cdot \log^\delta n}{n^{2-\sigma}} \left(\sigma(\sigma - 1) + \frac{\delta(2\sigma - 1)}{\log n} + \frac{\delta(\delta - 1)}{\log^2 n} \right) - \frac{g}{n^2} + o(1/n^2).$$

As we did above with the ratios, we eliminate the $O(1/n^2)$ term by constructing a modified ratio-of-ratios,

$$(18) \quad rr_n^{(2)} = \frac{n^2 rr_n^{(1)} - (n-1)^2 rr_{n-1}^{(1)}}{2n-1} = 1 + \frac{c \log^\delta n}{n^{2-\sigma}} (1 + O(1/\log n)),$$

where the constant $c = (\sigma^2(\sigma - 1) \log \kappa)/2$.

Then a plot of $\log |rr_n^{(2)} - 1|$ against $\log n$ should be close to linear, as the logarithmic term will vary very slowly over the range of n -values at our disposal, with gradient $\sigma - 2$. Such a plot (not shown) is visually linear, but in order to calculate the gradient we find the (local) gradient of the segment joining $rr_n^{(2)}$ and $rr_{n-1}^{(2)}$, which should approach the “correct” value as n increases. This is shown, plotted against $1/n$ in Figure 15. It appears to be going to a limit around -1.62 to -1.61 , which would imply $\sigma \approx 0.38$ or 0.39 , rather than the known value of $1/3$.

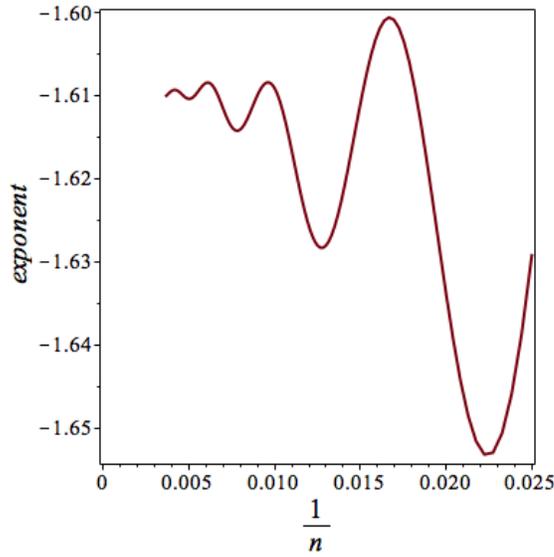


FIGURE 15. Estimators of exponent $\sigma - 2$ vs. $1/n$.

However, if we assume we know that $\delta = 2/3$, and include the confluent logarithmic term $\log^{2/3} n$ in the exponent of the stretched-exponential term, plotting instead

$$\log \left(\frac{r_n^{(2)} - 1}{\log^{2/3} n} \right)$$

against $\log n$, the plot is again visually linear. However the corresponding plot of the local gradient, shown in Figure 16, is clearly going to a limit around $-5/3$, consistent with the known value $\sigma = 1/3$.

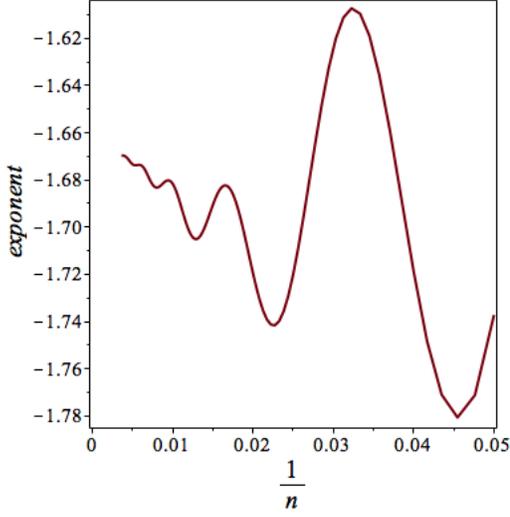


FIGURE 16. Estimators of exponent $\sigma - 2$ vs. $1/n$, assuming a confluent logarithmic term.

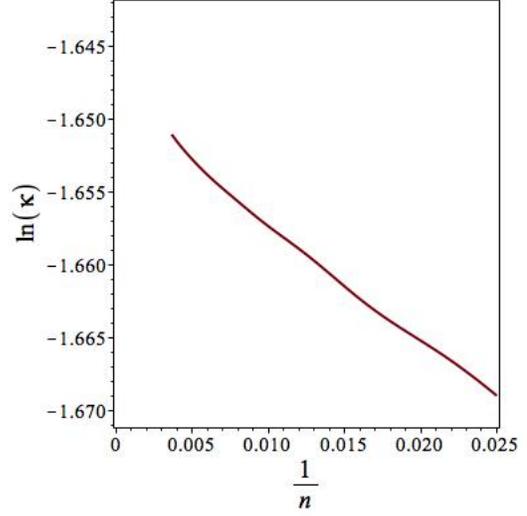


FIGURE 17. Estimates of $\log \kappa$ vs. $1/n$.

Assuming the values $\mu = 16$, and $\sigma = 1/3$ and $\kappa = 2/3$, we can estimate the remaining parameters in the asymptotic expression by direct fitting to the logarithm of the coefficients. From $c_n \sim c \cdot 16^n \cdot \kappa^{n^{1/3}} \log^{2/3} n \cdot n^g$ we get

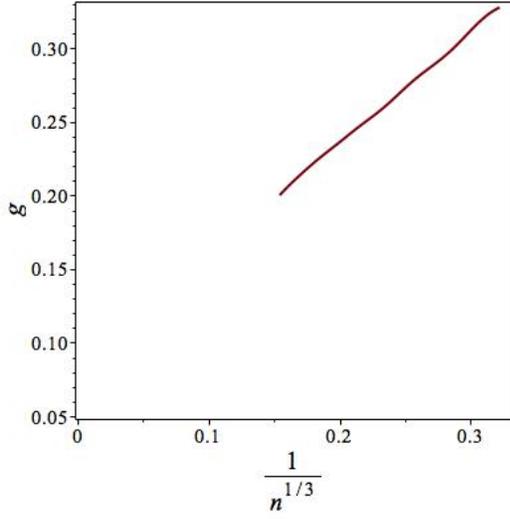
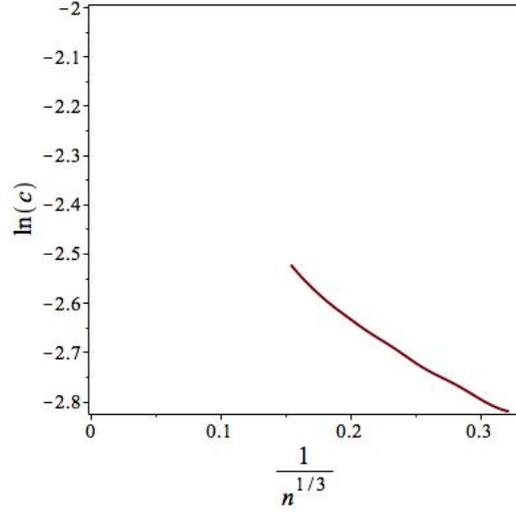
$$\log c_n - n \cdot \log 16 \sim n^{1/3} \cdot \log^{2/3} n \cdot \log \kappa + g \cdot \log n + \log c.$$

As in the preceding analysis of the lamplighter group coefficients, we fit successive triples of coefficients to get n -dependant estimates of the three unknowns, $\log \kappa$, g and $\log c$. The results are shown in Figures 17, 18, and 19 respectively.

From these plots, we estimate $\log \kappa \approx -1.64$, but it is difficult to estimate g . It appears to be quite small, close to zero, and could even be negative. It is even more difficult to extrapolate the plot for $\log c$, though one might conclude the bound $\log c \geq -2$. These estimates correspond to $\kappa \approx 0.194$, $g \approx 0$, and $c > 0.13$. As far as we are aware, these three constants have not previously been studied.

In anticipation of our analysis of Thompson's group F , where the growth constant μ is not known, we attempt to estimate both the exponents σ and δ without knowing the value of μ . Forming the ratios (15) eliminates the constant c in the asymptotic form of the coefficients, and the ratio of ratios (17) eliminates μ . If we now form the sequence

$$(19) \quad t_n = \frac{rr_n^{(1)} - 1}{rr_{n-1}^{(1)} - 1}$$


 FIGURE 18. Estimates of exponent g vs. $n^{-1/3}$.

 FIGURE 19. Estimates of $\log c$ vs. $n^{-1/3}$.

this eliminates the base κ of the stretched-exponential term, and in fact

$$n(t_n - 1) \sim \sigma - 2 + \frac{\delta}{n \log n}.$$

So plotting $n(t_n - 1)$ against $1/(n \log n)$ should give an estimate of $\sigma - 2$. To estimate δ , we form the sequence

$$n \log^2 n (n(t_n - 1) - (n - 1)(t_{n-1} - 1)) \sim -\delta + O(1/\log n).$$

We show these plots in Figures 20 and 21 respectively. The estimate of $\sigma - 2$ appears to be going to a limit of around -1.6 or below, c.f. the known exact value of $-5/3$, while the estimate of δ is harder to estimate, but the plot is certainly consistent with the known value $2/3$. As can be seen, this exponent is difficult to estimate without many more terms than we currently have.

5.5. Analysis of group $(\mathbb{Z} \wr \mathbb{Z}) \wr \mathbb{Z}$. For this group we have 132 terms in the cogrowth series, just less than half the number we have for $\mathbb{Z} \wr \mathbb{Z}$, so the results are not quite as precise. We analysed this series the same way as for the group $\mathbb{Z} \wr \mathbb{Z}$. For this group it is known that the coefficients grow exponentially, and that the dominant term is 36^n . The sub-dominant term is $\kappa^{n^{1/2} \log^{1/2} n}$, which again follows from Theorem 3.11 in [21]. Again, there is presumably a sub-sub dominant term n^g .

In this case we have for the ratio of successive terms:

$$(20) \quad r_n = \frac{a_n}{a_{n-1}} \sim \mu \left(1 + \frac{\log \kappa \log^{1/2} n}{2n^{1/2}} + \frac{\log \kappa}{2n^{1/2} \log^{1/2} n} + \frac{g}{n} + \dots \right).$$

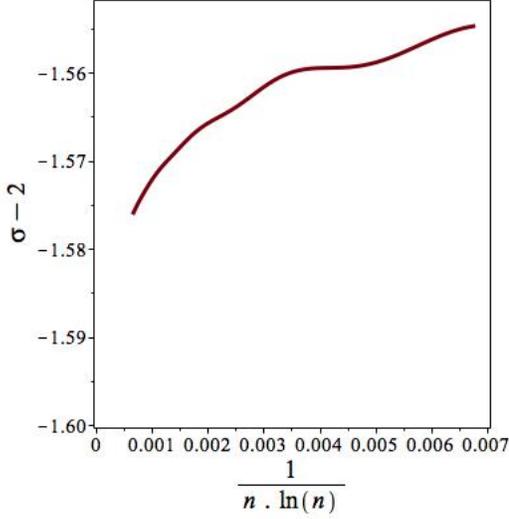


FIGURE 20. Estimates of $\sigma - 2$ vs. $1/(n \log n)$

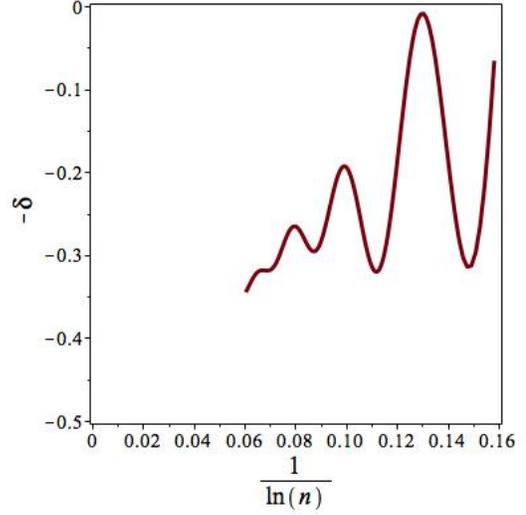


FIGURE 21. Estimates of exponent $-\delta$ vs. $1/\log n$.

We eliminate the $O(1/n)$ term by forming the modified ratios (14) which behave as

$$(21) \quad r_n^{(1)} = \mu \left(1 + \frac{\log \kappa}{4\sqrt{n}} \left(\sqrt{\log n} + 2 \log^{-1/2} n - \log^{-3/2} n \right) + o(n^{-3/2+\epsilon}) \right).$$

First, we remark that extrapolating the ratios against $1/n$ gives a plot with considerable curvature (not shown). We plotted the modified ratios, defined above, against $1/n^\sigma$ for several values of σ . We show the results for $\sigma = 1/2$ and $\sigma = 1/3$ in Figures 22 and 23 respectively. Surprisingly, the latter is closer to linear, however it extrapolates to a value of μ rather larger than the actual value, $\mu = 36$. However if we include the effect of the logarithmic term in the exponent, and plot (see equation (21)) the modified ratios against $\sqrt{\frac{\log n}{n}}$, the modified ratio plot, shown in Figure 24, is indistinguishable from linearity and extrapolates to the correct value of μ .

Repeating the analysis of the previous section, we attempted to estimate the exponent σ without assuming the value of the growth constant μ . A plot of $\log |rr_n^{(2)} - 1|$ (18) against $\log n$ should be close to linear, (as the logarithmic term will vary only slowly over the range of n -values at our disposal), with gradient $\sigma - 2$. Such a plot (not shown) is visually linear, but in order to calculate the gradient we find the (local) gradient of the segment joining $rr_n^{(2)}$ and $rr_{n-1}^{(2)}$, which should approach the ‘‘correct’’ value as n increases. This is shown, plotted against $1/n$ in Figure 25. It appears to be going to a limit below -1.42 which would imply $\sigma < 0.58$, compared to the known value of $1/2$.

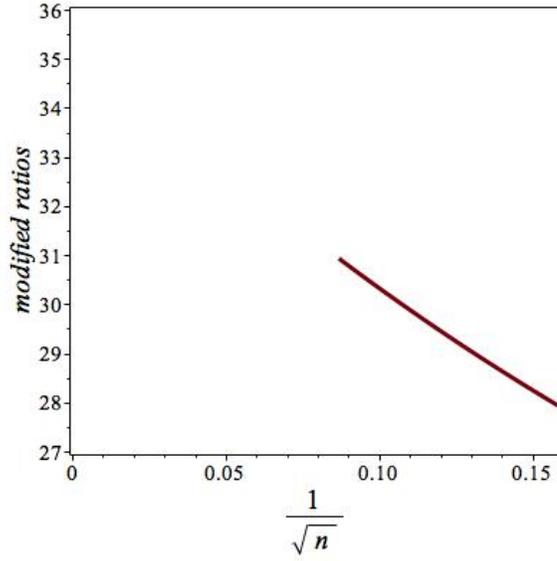


FIGURE 22. Modified ratios for $(\mathbb{Z} \wr \mathbb{Z}) \wr \mathbb{Z}$ vs. $1/\sqrt{n}$.

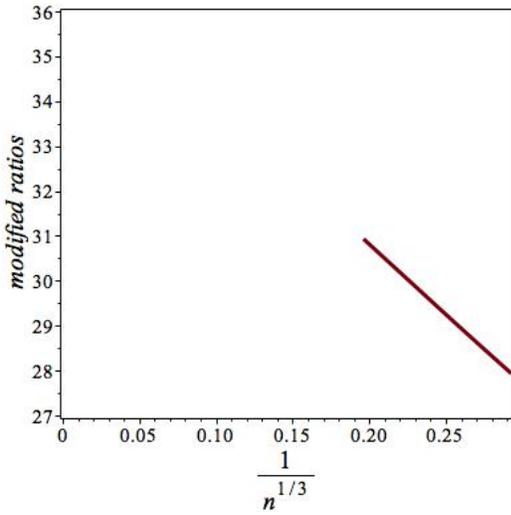


FIGURE 23. Modified ratios for $(\mathbb{Z} \wr \mathbb{Z}) \wr \mathbb{Z}$ vs. $n^{-1/3}$.

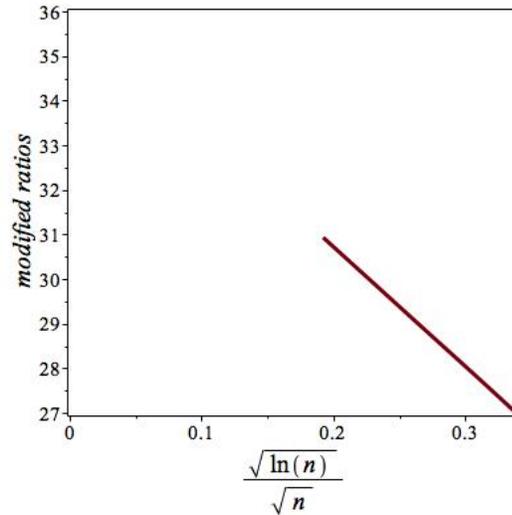


FIGURE 24. Modified ratios for $(\mathbb{Z} \wr \mathbb{Z}) \wr \mathbb{Z}$ vs. $\sqrt{\log n/n}$.

However, if we assume we know that $\delta = 1/2$, and include the confluent logarithmic term $\log^{1/2} n$ in the exponent of the stretched-exponential term, plotting instead

$$\log \left(\frac{r_n^{(2)} - 1}{\log^{1/2} n} \right)$$

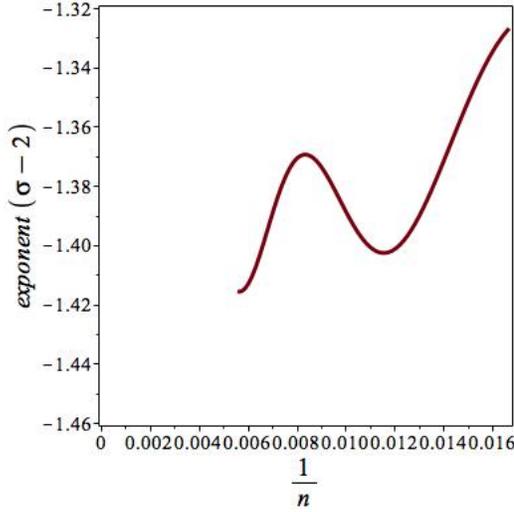


FIGURE 25. Estimators of exponent $\sigma - 2$ vs. $1/n$.

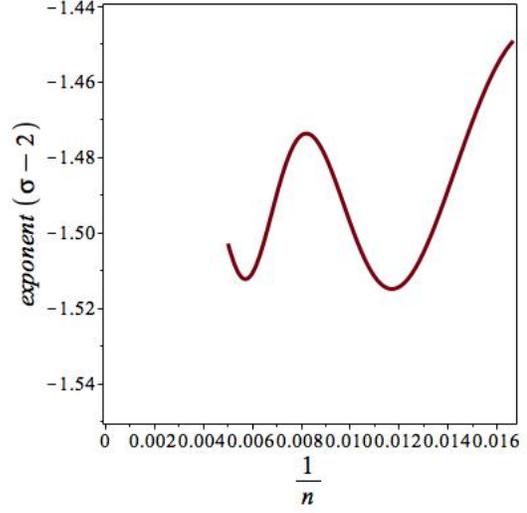


FIGURE 26. Estimators of exponent $\sigma - 2$ vs. $1/n$, assuming a confluent logarithmic term.

against $\log n$, the plot is again visually linear. Moreover the corresponding plot of the local gradient, shown in Figure 26, is going to a limit around $-3/2$, consistent with the known value $\sigma = 1/2$.

Assuming the values $\mu = 16$, $\sigma = 1/2$ and $\kappa = 1/2$, we can estimate the remaining parameters in the asymptotic expression by direct fitting to the logarithm of the coefficients. From $c_n \sim c \cdot 36^n \cdot \kappa^{n^{1/2} \log^{1/2} n} \cdot n^g$ we get

$$\log c_n - n \cdot \log 36 \sim n^{1/2} \cdot \log^{1/2} n \cdot \log \kappa + g \cdot \log n + \log c.$$

As in the preceding analysis of $\mathbb{Z} \wr \mathbb{Z}$, we fit successive triples of coefficients to get estimates of the three unknowns, $\log \kappa$, g and $\log c$. The results for the first two are shown in Figures 27 and 28 respectively. From this, and further analysis with an additional term in the assumed asymptotic form, we estimate $\log \kappa \approx -2.3$ and $g \approx 3.3$.

Again repeating the analysis of the previous section, we tried to estimate σ and δ directly without knowing μ or κ . Plotting $n(t_n - 1)$ (19) against $1/(n \log n)$ should give an estimate of $\sigma - 2$, and plotting the sequence $n \log^2 n (n(t_n - 1) - (n - 1)(t_{n-1} - 1))$ against $1/\log n$ should give estimates of $-\delta$. We show these plots in Figures 29 and 30 respectively. The estimate of $\sigma - 2$ appears to be going to a limit of below -1.39 or so, c.f. the known exact value of -1.5 , while it is not possible to estimate δ from this plot, but it is not inconsistent with the known value $1/2$.

5.6. The group $\mathbb{Z} \wr_d \mathbb{Z}$. In the previous sections we have considered the analysis of the groups $\mathbb{Z} \wr_d \mathbb{Z}$ for $d = 1$ and $d = 2$. We have shown how the stretched-exponential term slows the rate of convergence of the ratios, but that appropriate analysis can still reveal

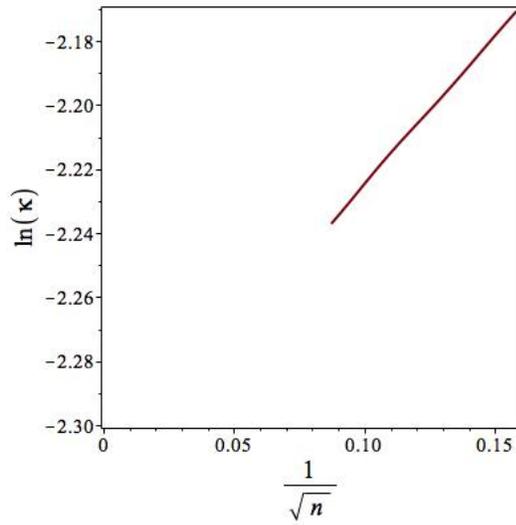


FIGURE 27. Estimates of $\log \kappa$ vs. $1/\sqrt{n}$.

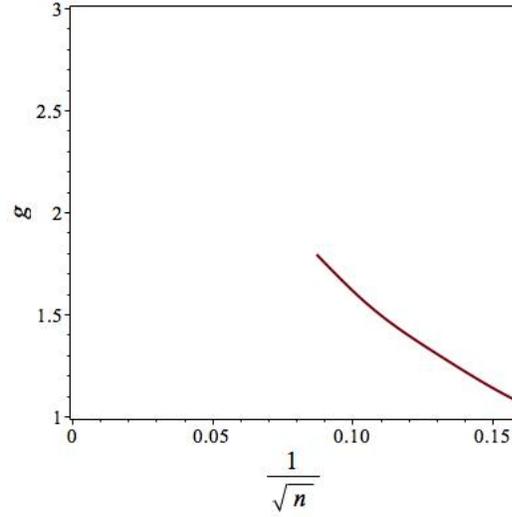


FIGURE 28. Estimates of exponent g vs. $1/\sqrt{n}$.

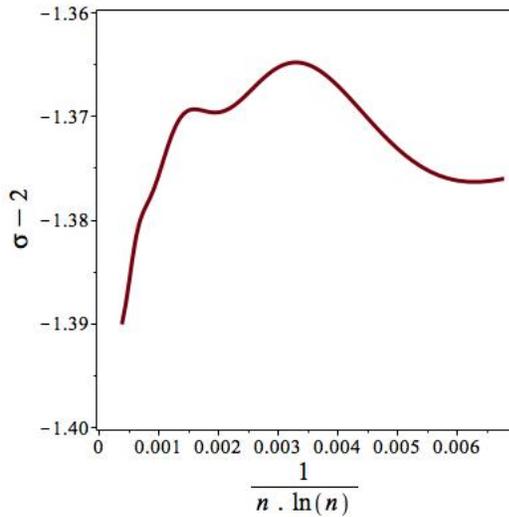


FIGURE 29. Estimates of $\sigma - 2$ for $(\mathbb{Z} \wr \mathbb{Z}) \wr \mathbb{Z}$ vs. $1/(n \log n)$

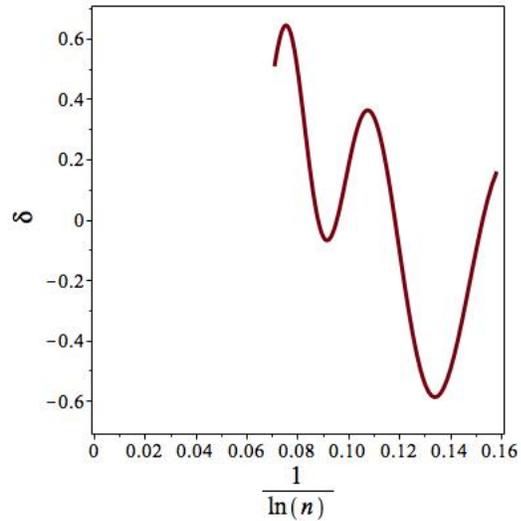


FIGURE 30. Estimates of exponent δ for $(\mathbb{Z} \wr \mathbb{Z}) \wr \mathbb{Z}$ vs. $1/\log n$.

much asymptotic information. However as d increases, it becomes increasingly difficult to extract the asymptotics from a hundred or so terms of the cogrowth series. To see this, we

consider the case $d = 98$. Then we know the asymptotic form of the coefficients is

$$c_n \sim c \cdot \mu^n \cdot \kappa^{n^\sigma \log^\delta n} \cdot n^g,$$

where $\sigma = 49/50$ and $\delta = 1/50$ [21].

While we could have generated 100 or so terms of this series from the algorithms described above, it will be more instructive to generate a test series with the given asymptotic behaviour, as then we can generate thousands of terms essentially immediately.

So we have generated coefficients defined by $c_n = c \cdot \mu^n \cdot \kappa^{n^\sigma \log^\delta n} \cdot n^g$ with $c = 1$, $\mu = 4$, $\kappa = 0.7$, $g = 0.5$, $\sigma = 49/50$ and $\delta = 1/50$. The ratio of successive terms must go to 4.0, the value of the growth constant⁴. Using 128 terms of this test series, we show a plot of the ratios against $1/n$ in Figure 31. It is not possible to assert that, as $n \rightarrow \infty$ the ratios will go to 4.0. In Figure 32 we show the same plot with 1280 terms. While this curve is steeply increasing, it is still not possible to assert that the limiting value is 4.0. Using 10000 terms, and plotting the ratios against $1/n^{1/50}$ (not shown), we finally see evidence that the extrapolated limit is around 3.8 or 3.9.

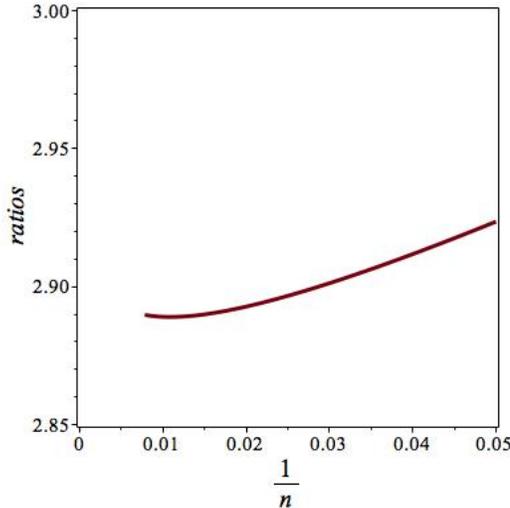


FIGURE 31. The first 128 ratios for $\mathbb{Z}_{\lambda_{98}} \mathbb{Z}$ vs. $1/n$.

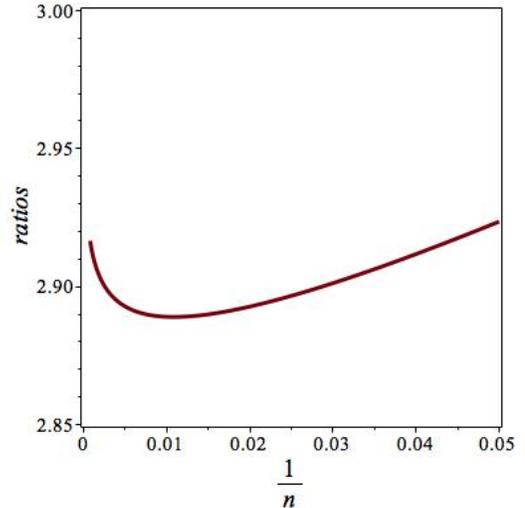


FIGURE 32. The first 1280 ratios for $\mathbb{Z}_{\lambda_{98}} \mathbb{Z}$ vs. $1/n$.

For this series the asymptotic form of the ratios is

$$r_n = \mu \left(1 + \frac{49 \log \kappa}{50 \cdot n^{1/50}} + \frac{0.5}{n} + o(1/n) \right),$$

so we might expect more informative results if we eliminate the term $O(1/n)$, which we can do by forming the modified ratios. These are shown, plotted against $1/n^{1/50}$ in Figures

⁴The growth constant is actually $4(d+1)^2$, but for this exercise the actual value is irrelevant, so we have chosen a much smaller value.

33 and 34, based on the first 128 terms and the first 10000 terms. Extrapolating these to $n \rightarrow \infty$ again gives a limit around 3.9.

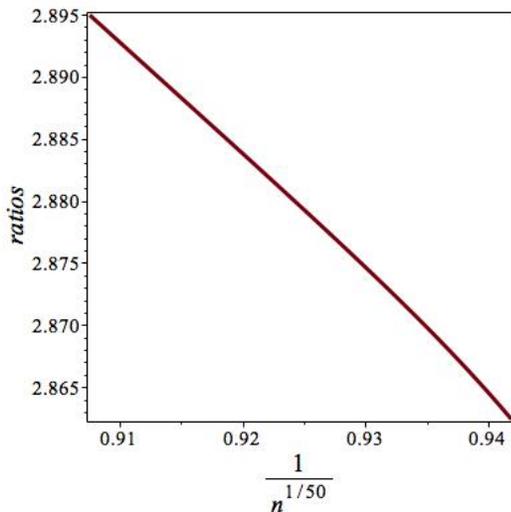


FIGURE 33. The first 128 modified ratios for $\mathbb{Z} \wr_{98} \mathbb{Z}$ vs. $n^{-1/50}$.

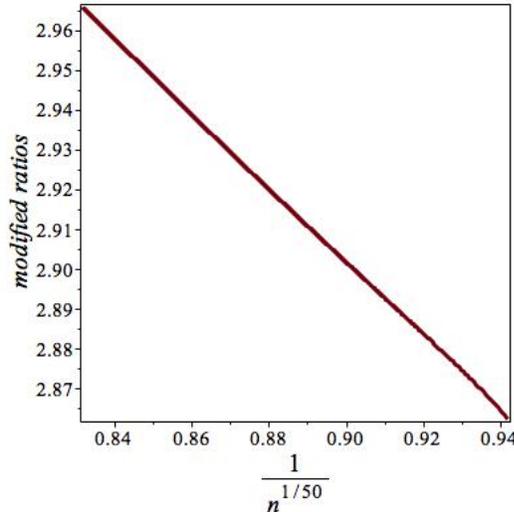


FIGURE 34. The first 10000 modified ratios for $\mathbb{Z} \wr_{98} \mathbb{Z}$ vs. $n^{-1/50}$.

It is possible to estimate the exponent σ without knowing μ , as we showed in previous examples above. In particular, using the method based on equation (17), and described immediately below that equation, we show in Figure 35 a plot of estimators of $\sigma - 2$ against $1/n$, based on a 10000 term series, and it is persuasively going to the known value -1.02 .

Unfortunately, for no interesting problem is it realistic to get 10000 terms, so this example, and the next, must remain as a cautionary tale, to the extent that there can and do exist groups whose cogrowth series exhibit asymptotic behaviour that is difficult to estimate by numerical methods of the type we have considered. Another example of similar difficulty is given by the Navas-Brin group B , discussed in the next section.

6. SERIES EXTENSION

In this section we develop one further tool that will be extremely useful in our analysis of the series for Thompson's group F , where we have only 32 terms, rather than a hundred or more as in the examples we have been considering. It will also be very helpful in our analysis of the Navas-Brin group B , discussed in the next section.

Recall that our analysis of the more complex asymptotic forms that include stretched-exponential terms is based on ratios of successive terms, whereas for simpler groups, with simpler asymptotics, we used the method of differential approximants (DAs). It is obviously highly desirable to have further terms (in particular, further ratios), for all series with non-simple asymptotics, and particularly in those cases where we have comparatively short

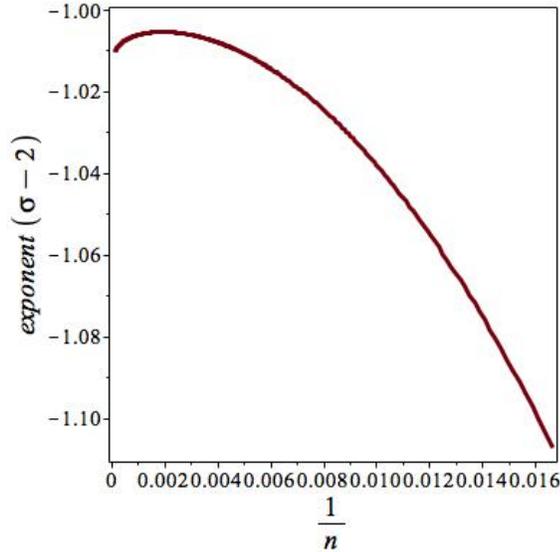


FIGURE 35. Estimators of $\sigma - 2$ for $\mathbb{Z}_{l_{98}}\mathbb{Z}$ against $1/n$ for $n \leq 10000$.

series, such as the 32 term series we have for Thompson's group F . In order to obtain further ratios (or terms), we use the method of differential approximants *to predict subsequent ratios/terms*. The detailed description as to how this is done is given in [15].

We will give two demonstrations of the effectiveness of this method. In the first, we take the first 32 terms of the series for $\mathbb{Z}l\mathbb{Z}$ discussed above, (we have more than 200 terms for this series), and use these to predict the next 89 ratios, from 5th order DAs. As well as the mean ratio, we calculate the standard deviation. We show, in Table 3, a comparison between the actual error in the predicted ratios and the standard deviation of the estimated ratios. It can be seen that the true error lies between 1 and 1.5 standard deviations, which provides some confidence that the predicted ratios are accurate to within an error of 1.5 standard deviations.

For the series simulating the coefficients of the group $\mathbb{Z}_{l_{98}}\mathbb{Z}$, we showed the importance of long series to reveal the asymptotic behaviour with some precision. In this second example, we take the first 100 terms of this series, and use them to predict the *next 315 ratios*. That is, we estimate c_n/c_{n-1} for $n = 101 \cdots 415$.

To see how precisely these ratios can be predicted, we plot the difference between the actual ratios and those calculated by 4th order differential approximants in Figure 36. It can be seen that the error is less than 2 parts in 10^{20} for all $n < 416$. Just to make this perfectly clear, given 100 coefficients, we have predicted the next 315 ratios with an accuracy of some 20 significant digits.

In a similar fashion, using 4th order DAs, we were able to get 200 extra ratios for the 32-term series for Thompson group F . The maximum error (as estimated by 1.5 s.d. of the

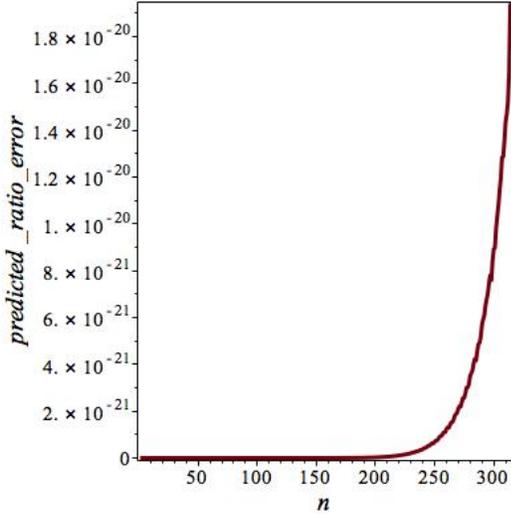


FIGURE 36. Absolute error in predicted ratios of $\mathbb{Z} \wr_{98} \mathbb{Z}$ for $100 < n < 416$.

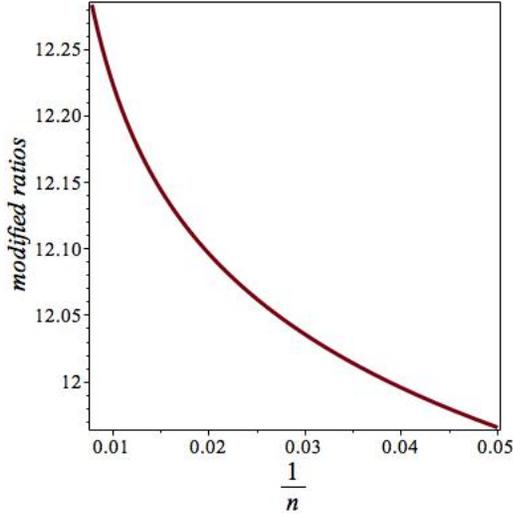


FIGURE 37. The first 128 modified ratios of the Navas-Brin group B vs. $1/n$.

k	Actual error	1 standard deviation
1	2.69×10^{-17}	2.02×10^{-17}
5	1.14×10^{-13}	7.85×10^{-14}
10	3.37×10^{-11}	2.08×10^{-11}
20	2.22×10^{-8}	1.23×10^{-8}
30	9.63×10^{-7}	5.39×10^{-7}
40	1.22×10^{-5}	6.88×10^{-6}
50	7.59×10^{-5}	4.73×10^{-5}
60	3.13×10^{-4}	2.23×10^{-4}
70	9.39×10^{-4}	8.11×10^{-4}
80	2.44×10^{-3}	2.44×10^{-3}
89	4.63×10^{-3}	5.38×10^{-3}

TABLE 3. Actual error in coefficient $O(z^{31+k})$ and 1 standard deviation from the mean of the estimated coefficient.

DAs) is 1 part in 4×10^{-5} , which is graphically imperceptible. In the Appendix we give the (predicted) next 200 ratios, and their standard deviations.

7. ANALYSIS OF THE NAVAS-BRIN GROUP B .

This is an amenable group introduced independently by Navas [19] and Brin [4], so we call it the Navas-Brin group B , and is defined in subsection 2.2. It has 2 generators, so

the growth rate of the cogrowth sequence is 16. We gave a polynomial-time algorithm to generate the coefficients above, and have used this to generate 128 terms of the co-growth series. We then used the method of series extension, described above, to give a further 590 ratios, the last of which we expect to be accurate to 1 part in 5×10^{-7} , while all earlier ratios will have a lower associated error. We first show a plot of the modified ratios (14) against $1/n$ in Figure 37. Even if we knew nothing about the asymptotics of this group, the curvature of this plot provides strong evidence for a sub-exponential term, and we have proved that it cannot be a regular stretched-exponential term.

That is to say, the asymptotics for this series must grow more slowly than

$$c_n \sim c \cdot \mu^n \cdot \kappa^{n^\sigma} \cdot n^g,$$

where $\mu = 16$, $0 < \sigma < 1$, and $0 < \kappa < 1$. Possible behaviour might be

$$c_n \sim c \cdot \mu^n \cdot \kappa^{n/\log n} \cdot n^g,$$

corresponding to a *numerical* value $\sigma = 1$, which of course hides the logarithmic component.

In that case the ratios will be

$$r_n = \frac{c_n}{c_{n-1}} \sim \mu \left(1 + \frac{\text{constant}}{\log n} + \frac{g}{n} + \dots \right).$$

Note that we do not insist the the first correction term is $O(1/\log n)$, it could be a power of a logarithm, or some other weakly decreasing function, but it cannot have a power-law increase. For our purposes it suffices to take this term to be $O(1/\log n)$. We show the modified ratios (this gets rid of the $O(1/n)$ term in the asymptotics) in Figures 38 and 39 which are the same plot, but the first uses only the 128 exact coefficients, while the second uses the exact plus predicted ratios. From the first plot, it is clear that it would be an article of faith that the locus is going to 16 as $n \rightarrow \infty$. By contrast, the second plot makes this conclusion far more plausible.

We next try and estimate the exponent σ , which should be 1, without assuming $\mu = 16$. We use the method described below equation (17). With the 128 known terms, the estimators of $\sigma - 2$ are shown in Figure 40 and show no evidence of approaching the expected value of -1 . If however we use twice as many terms, so using the next 128 predicted ratios, we get the plot shown in Figure 41, which *is* plausibly approaching -1 .

This highlights the value of numerically predicting further terms wherever possible.

8. ANALYSIS OF THOMPSON'S GROUP F

For Thompson's group F it is known that the series grows exponentially like μ^n . If $\mu = 16$, the group is amenable. If it is amenable, there cannot be a sub-dominant term of the form κ^{n^σ} with $0 < \sigma < 1$, because the group contains the wreath products $\mathbb{Z} \wr \mathbb{Z} \wr \mathbb{Z} \wr \dots \wr \mathbb{Z}$ as subgroups. This is a consequence of Theorem 1.3 in [20] and results in [21], and is proved as Theorem 3.2 in Section 3.

We first study the modified ratios, defined by (14). The modified ratio plot against $1/n$ is shown in figure 42 and displays considerable curvature. By contrast, the same data plotted against $n^{-1/5}$, and shown in figure 43 shows curvature in the opposite direction.

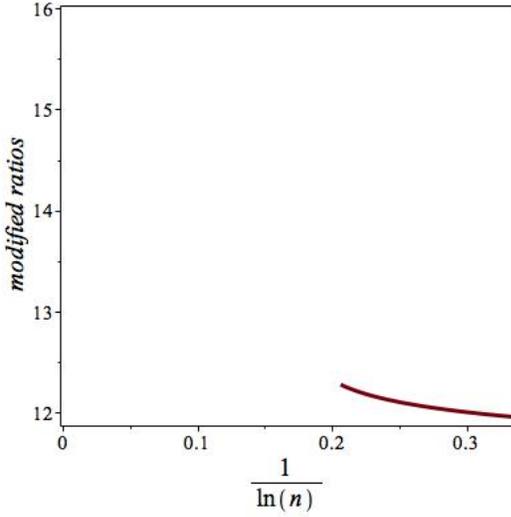


FIGURE 38. The first 128 modified ratios for the Navas-Brin group B vs. $1/\log n$.

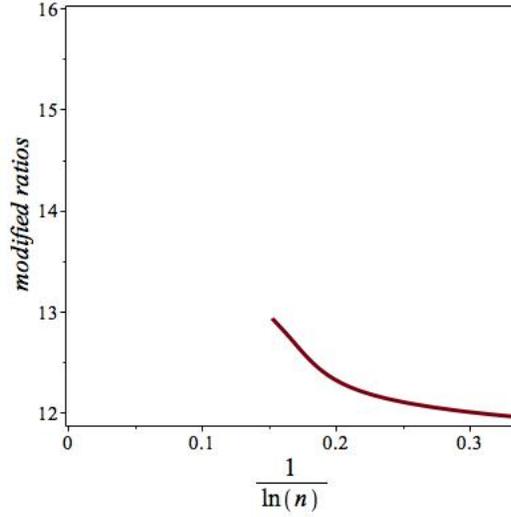


FIGURE 39. The first 718 modified ratios for the Navas-Brin group B vs. $1/\log n$.

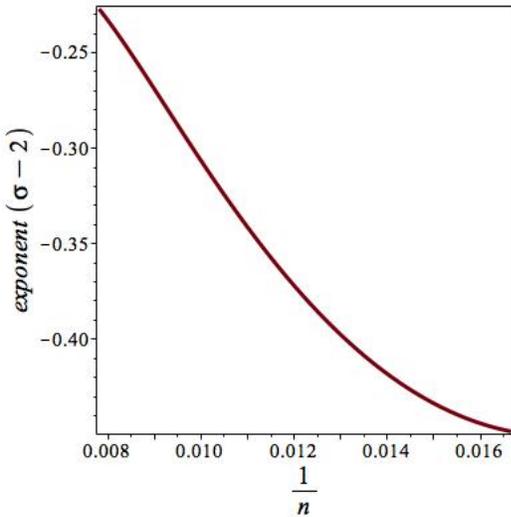


FIGURE 40. Estimates of $\sigma - 2$ from 128 terms of the Navas-Brin group B .

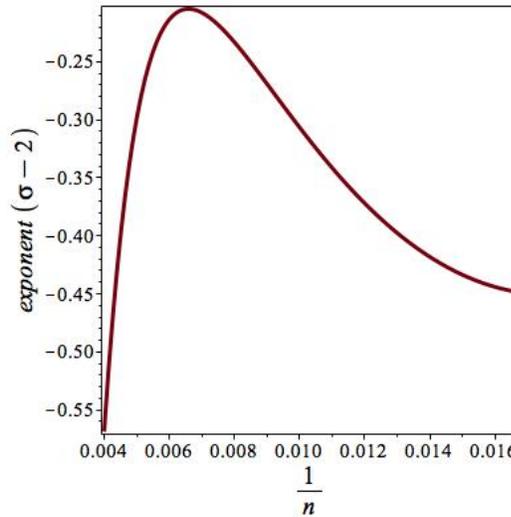


FIGURE 41. Estimates of $\sigma - 2$ from 256 terms of the Navas-Brin group B .

This is strong evidence for the presence of a conventional stretched-exponential term of the sort we have seen in our study of the lamplighter group and the family W_d . As mentioned above, the presence of such a term is incompatible with amenability. This is our first piece

of evidence that the group is not amenable. Note too that this is quite different to the behaviour observed for the coefficients of the Navas-Brin group B .

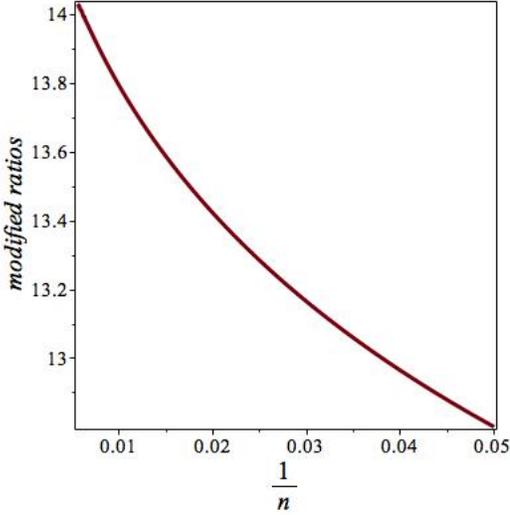


FIGURE 42. Modified ratios vs. $1/n$ for Thompson's group F .

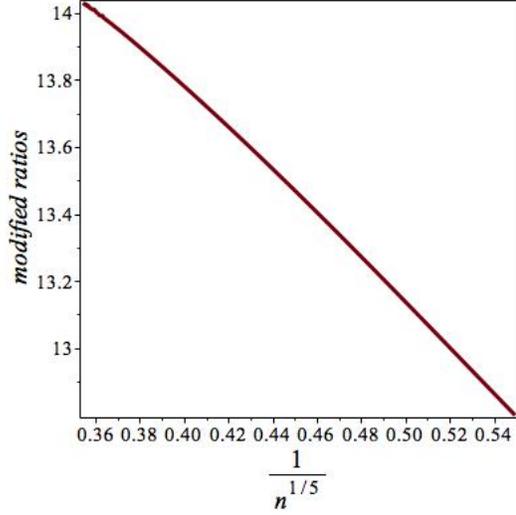


FIGURE 43. Modified ratios vs. $n^{-1/5}$ for Thompson's group F .

In our subsequent analysis, we use both the exact coefficients and the extrapolated coefficients. While all extrapolated terms can be used in calculating the ratios, once one calculates first and second differences, errors are amplified, and so fewer terms can be used. That is why we quote the number of terms used for different calculations, as it is only to the quoted order that we are confident that the calculated quantities are accurate to graphical accuracy.

To estimate the exponents in the stretched-exponential term we use the procedure described in Section 5.4, given by eqn. (19) and subsequent equations. This procedure allows for the presence of a confluent power of a logarithm, so that the stretched-exponential term is $\kappa^{n^\sigma \log^\delta n}$. In this way, based on a series of length 80, we show plots of estimators of $2 - \sigma$ and $-\delta$ in Figures 44 and 45, plotted against $1/n$. Extrapolating these, we estimate $\sigma \approx 1/2$, and $\delta \approx 1/2$. Recall that this is exactly the stretched-exponential behaviour of $(\mathbb{Z} \wr \mathbb{Z}) \wr \mathbb{Z}$.

Reverting to the modified ratios, briefly discussed above, we plot these against $1/\sqrt{n}$ in figure 46, using 186 terms. One observes that the plot still displays a little curvature, but in Figure 47 the plot of these same modified ratios against $\sqrt{\log n}/n$, is essentially linear. This is the appropriate power to extrapolate against, given our estimates of the stretched-exponential exponents. Extrapolating this to $n \rightarrow \infty$ we estimate the limit, which gives the growth constant, to be 14.8 – 15.1. This is well away from 16, which would be required for amenability.

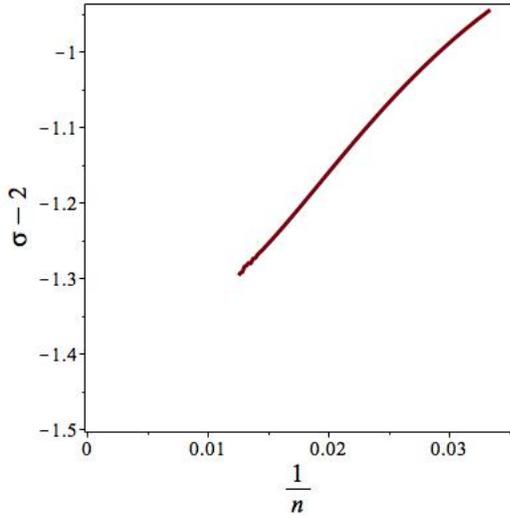


FIGURE 44. Estimators of $\sigma - 2$ for Thompson's group F vs. $1/n$.

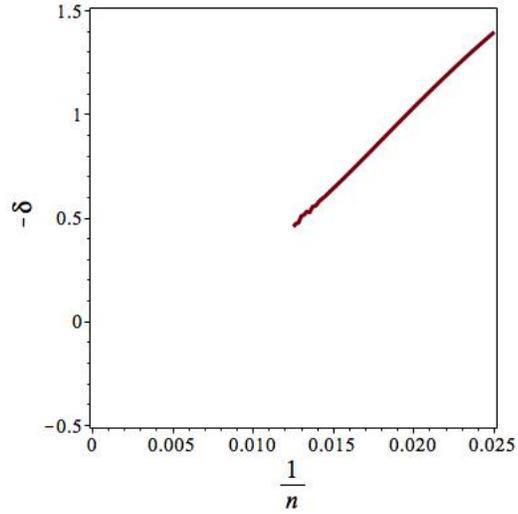


FIGURE 45. Estimators of $-\delta$ for Thompson's group F vs. $1/n$.

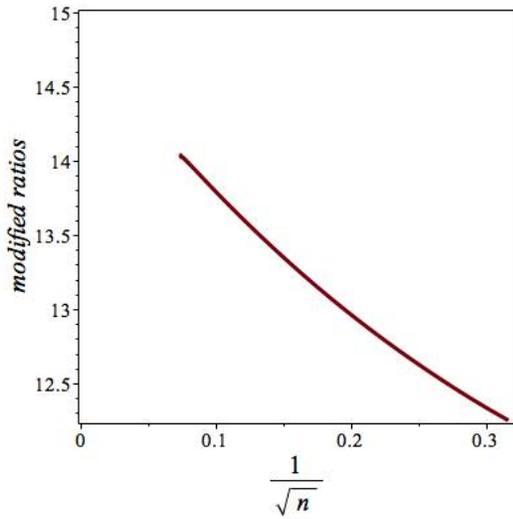


FIGURE 46. The first 186 modified ratios for Thompson's group F vs. $1/\sqrt{n}$.

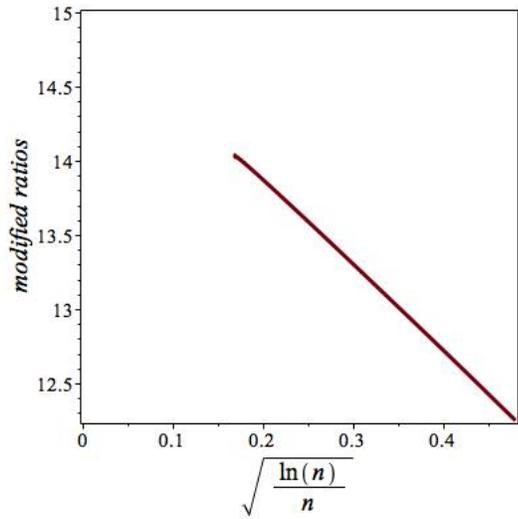


FIGURE 47. The first 186 modified ratios for Thompson's group F vs. $\sqrt{\log n/n}$.

One simple test for amenability uses the fact that the ratio of successive coefficients asymptotes to the growth constant μ . For the lamplighter group, this ratio behaves as

$$r_n^{(L)} = 9 \left(1 + \frac{c}{n^{2/3}} + o\left(\frac{1}{n^{2/3}}\right) \right).$$

For $\mathbb{Z} \wr \mathbb{Z}$ one has

$$r_n^{(2)} = 16 \left(1 + \frac{c \cdot \log^{2/3} n}{n^{2/3}} + o\left(\frac{\log^{2/3} n}{n^{2/3}}\right) \right),$$

and for the triple wreath product, W_2 , the corresponding result is

$$r_n^{(3)} = 36 \left(1 + \frac{c\sqrt{\log n}}{n^{1/2}} + o\left(\frac{\sqrt{\log n}}{n^{1/2}}\right) \right),$$

while for Thompson's group F all we know is

$$r_n = \mu (1 + \text{lower order terms}),$$

where we suspect that the correction term is similar to that of the triple wreath product of \mathbb{Z} .

So, a simple test for amenability is to look at the three quotients

$$\frac{9r_n}{16r_n^{(L)}}, \quad \frac{r_n}{r_n^{(2)}}, \quad \text{and} \quad \frac{4r_n}{9r_n^{(3)}}.$$

If Thompson's group F is amenable, these quotients should all go to 1. In Figures 48, 49, 50 we show these ratios plotted against $\sqrt{\log n/n}$, which is the appropriate power, though this choice is not critical. The ratios do not appear to be going to 1 in any of the three cases. For all cases we have used 200 ratios. To do this, we used the extended ratios for Thompson's group F and also extended the ratios for W_2 from the known 132 ratios. Indeed, all three cases are consistent with a limit around 0.93 ± 0.02 , corresponding to $\mu = 14.9 \pm 0.3$. This is entirely consistent with our previous estimate of $\mu \approx 15.0$.

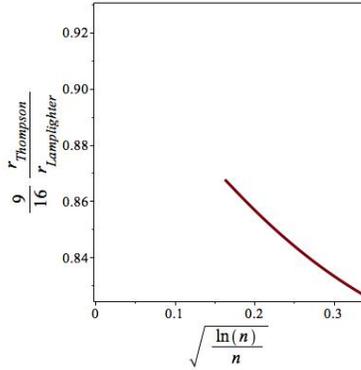


FIGURE 48. Quotient of Thompson group and lamplighter group ratios using 200 terms.

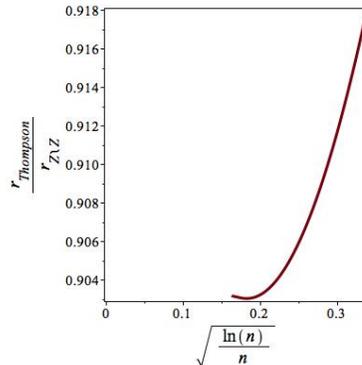


FIGURE 49. Quotient of Thompson group and $\mathbb{Z} \wr \mathbb{Z}$ ratios using 200 terms.

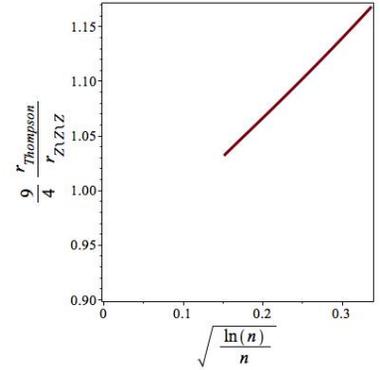


FIGURE 50. Quotient of Thompson group and $(\mathbb{Z} \wr \mathbb{Z}) \wr \mathbb{Z}$ ratios using 200 terms.

Finally, we take the approach of extrapolating the lower bounds produced in Section 4. Note that the sequence of bounds $\{b_n\}$ are bounds on $\sqrt{\mu}$. We have no expectation as to

how this sequence should approach its limit, so we first plot the bounds against $1/n$ in figure 51. Some curvature is seen, which, as we have shown above, is evidence that the locus behaves as

$$b_n \sim b_\infty \left(1 + \frac{c_1}{n^\alpha} + \frac{c_2}{n} + \dots\right).$$

We remove the term $O(1/n)$ in this case by forming the sequence $b_n^{(1)} = (n \cdot b_n - (n-2) \cdot b_{n-2})/2$ where we have shifted n by 2 to remove the effect of a small odd-even oscillation if one shifts only by 1. We found that plotting $b_n^{(1)}$ against $1/\sqrt{n}$ gave a visually linear plot, and this is shown in figure 52. Linearly extrapolating the last two entries gives the estimate $b_\infty \approx 3.875$, so that $\mu \approx 15.02$, in agreement with previous estimates above.

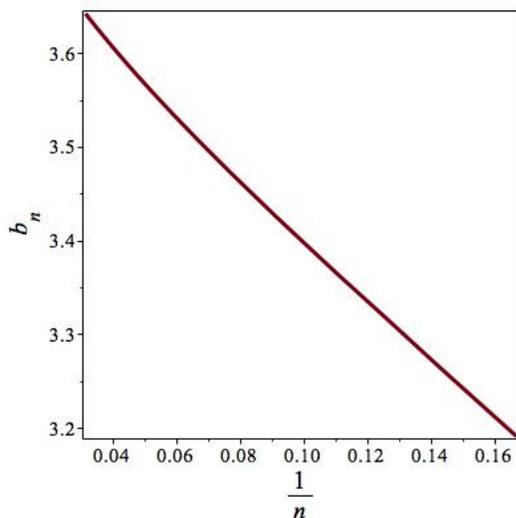


FIGURE 51. Plot of bounds b_n for Thompson's group F against $1/n$.

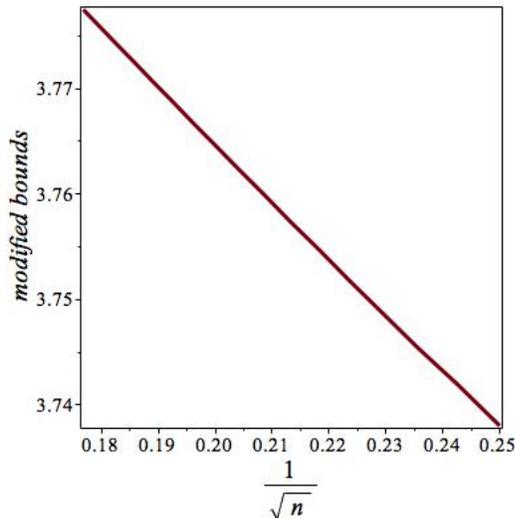


FIGURE 52. Plot of modified bounds $b_n^{(1)}$ for Thompson's group F against $1/\sqrt{n}$.

9. CONCLUSION

We have given polynomial-time algorithms to generate terms of the cogrowth series for several groups. In particular, we have given the first series for the Navas-Brin group B . We have also given an improved algorithm for the coefficients of Thompson's group F , giving 32 terms of the cogrowth series, extending previous enumerations by 7 terms. We analysed these various series to develop numerical techniques to extract the asymptotics, and gave improved asymptotics for the Heisenberg group. We gave an improved lower bound on the growth-rate of the cogrowth series for Thompson's group F , $\mu \geq 13.2693$ using the method in [18]. We generalised their method, showing that the cogrowth sequences for all these groups can be represented as the moments of a distribution. Extrapolation of the sequence of bounds suggests the limit is around 15.0, which is incompatible with amenability.

For Thompson's group F we proved that, if the group is amenable, there cannot be a sub-dominant stretched exponential term in the asymptotics. The numerical data however provides compelling evidence for the presence of such a term. This observation suggests a potential path to a proof of non-amenability.

We have extended the sequence of 32 terms for group F by a further 200 terms (or, as appropriate, 200 ratios of successive terms), which we demonstrate are sufficiently accurate for the graphical approaches to analysis that we have taken.

A numerical study of the cogrowth sequence c_n gives

$$c_n \sim c \cdot \mu^n \cdot \kappa^{n^\sigma \log^\delta n} \cdot n^g,$$

where $\mu \approx 15$, $\kappa \approx 1/e$, $\sigma \approx 1/2$, $\delta \approx 1/2$, and $g \approx -1$. The growth constant μ must be 16 for amenability. This estimate of the growth constant is the same as that obtained from the extrapolated bounds. These three approaches to the study of amenability lead us to the strong belief that Thompson's group F is not amenable.

The difficulties we encountered in analysing $\mathcal{Z}_{\log \mathcal{Z}}$ and the Navas-Brin group B does imply that there do exist groups whose cogrowth series are difficult to analyse. Nevertheless, in both those cases we were able to extract the correct asymptotics. Furthermore, the cogrowth series for Thompson's group F did not behave like either of these two "difficult" groups, and indeed appeared to have a stretched exponential term with exponent values that were readily estimable. While we cannot rule out the presence of some previously unsuspected pathology in the asymptotic form, we believe that we have presented strong evidence for the belief that Thompson's group F is not amenable

10. APPENDIX

Here are the (predicted) next 200 ratios for Thompson's group F . That is, the first ratio here is the coefficient of z^{32} divided by the coefficient of z^{31} . One standard deviation is 1.6×10^{-21} for the first ratio, 8.4×10^{-16} for the tenth ratio in this list, then 2.2×10^{-13} , 3.8×10^{-11} , 8.3×10^{-10} , 7.5×10^{-9} , 3.1×10^{-8} , 3.5×10^{-7} , 1.2×10^{-6} , 5.3×10^{-6} , 3.3×10^{-5} , 9.0×10^{-5} , 1.3×10^{-4} , for the twentieth, thirtieth, fortieth, fiftieth, seventieth, ninetieth and hundred and tenth, hundred and thirtieth, hundred and fiftieth, hundred and seventy-fifth and two hundredth ratios, respectively.

12.139382519134640546100910550116506	12.16995235080083581883533877031972	12.199326127345853916009149880943422
12.227584513675824849745149961326117	12.254800541517346423861272221078461	12.281040527431431456883217428846113
12.306364858371755791208652657890394	12.330828666879163731631451465102197	12.354482413853570301131793493786704
12.377372393555580912478863352420963	12.399541172868336566805611620482939	12.421027974747801610713466511114207
12.441869014094574198504147444796094	12.462097792906309126366784632966479	12.481745360450139667192011174605863
12.500840543278316369259845344418039	12.519410149156226556638850314986873	12.537479148348791828180725386201064
12.555070835196117213101528584412301	12.572206972475257876321270606856264	12.588907920692053314139061768312908
12.605192754131413602669204643768679	12.621079365262119276409607772782102	12.636584558849435543137582196072181
12.651724136967118435759923721500215	12.666512975937091192133384667614540	12.680965096098118923436248549296051
12.695093725180061251425934047225685	12.708911355958942268539084708497350	12.722429798828036178804146087289492
12.735660229810914461533323666877360	12.748613234321123627064424451294341	12.761298847540600696316471357098745
12.773726591097905739200796016562265	12.785905507056272163479082083260714	12.797844188914176808112395397996995
12.809550810208112073629601806420114	12.8210335151370240355220794516217934	12.832298623805456848288841282862554
12.843354291690656209817237940576090	12.854206894770548244369354384398327	12.864862864837654839427475941273709
12.875328347113943574400174300355914	12.88560921474516285655480222623966	12.89571108516949358938671835788695
12.9056393321510850772001331719253254	12.915399101706732471897030527017181	12.924995323677800300649200885376963
12.9344327239227972410562601010730083	12.9437158393802561160942396761922280	12.952849018349319276368496909220240
12.961836439248190289410424756818502	12.970682107319771206377319240620875	12.979389881806587929393146535001536
12.987963464694516602362349323441444	12.996406405676872159827925022024219	13.004722173676474514325001997839461
13.012914074054692135066857458367114	13.020985259779764800684267751152161	13.028938801180381908385125175110548
13.03677659579423368614400305996310	13.044504679013068113321447729698093	13.052122569524550318903068986491322
13.059634066593784780607389266957567	13.067041693422999439936815568787524	13.074347966865955839456185038291272
13.08155522866179498657087709047136	13.088665732145507263332155324286416	13.095681797656212049861838152969811
13.102605564520384538741633073251136	13.109439117936998695729698444879843	13.116184350936469838871049112592397
13.12284346766105221853487988958406	13.129418244175823190427250023546405	13.135910414942072482775413880736439
13.14232234451185942341571946995161	13.148654712658935599802876138058446	13.154910018341609905494554223902216
13.161089481701929123360329350075889	13.167195007769142327138772890144629	13.173227515058212755159290331997399
13.179189107961895176661932566775513	13.185080837993801908011934614407469	13.190904563910270596599158822879997
13.196660991623399487925223492990383	13.202352151456208348963284966653968	13.207979050670762502542808795553103
13.213542543992015500848473483969209	13.219044110867187297414288617379158	13.224484921862603128812749940479344
13.229866114857320555465055797943269	13.235188321357468124261384585429120	13.240453522986667992419868738673504
13.245661794080940364573430988057957	13.250815166741899371063964386311321	13.25591414765153334535516275030551
13.260959939896671980546840767426691	13.265952758167346821144011231194731	13.270895130906341393601998277628536
13.275786005549024172548265832401768	13.280627056720430964223829034834261	13.285419125190171039873138415974114
13.290161972217217703156676577608563	13.294858653114477361638435828704210	13.299508767603123209430266638744025
13.304113073325880062519253728849004	13.308670446314943572746000301996392	13.313185140342015149024233058130128
13.31765818828211633526537329472658	13.322086115777917425146418299128619	13.326471695739198888194127423197993
13.330818453996063928243407490059108	13.335121539230967797838900373066820	13.339384163779883682357243658284554
13.343606937084326659382677923429016	13.34778632027756923575569080786819	13.351930748190988263794674502771721
13.35604186569912853306677923821716	13.360110874363270214471235529540315	13.364142777237566574247813605863778
13.368138079655518061737105379768840	13.372097273034426235102779701961991	13.376020835131593723786616298972917
13.379909230285796422796584282677189	13.383762909644345312962700551534152	13.387582311376027194727073062599142
13.391391013985309598551994527199420	13.395144666969365978147066024081715	13.398865377039202234681959848510147
13.402540990852287889254736126437580	13.406196253710405879515140382321201	13.409819665483104102788219559296390
13.413411407143425386912606665853825	13.41698815657036706622724319777381	13.420539883769483957581690933051480
13.424062672750839568072161252780645	13.427535999385894978339537743097391	13.430979699159296367928654845659964
13.434394074827496721484051571897707	13.437779418859588380431049420003005	13.441136013513055381428134566110019
13.44448284312927251038029918715121	13.447782268650965022269207427038709	13.451072044972393249466607491105591
13.454316759036914691255964338022543	13.457544856354882926610701767183908	13.460737144623883340126835494564910
13.463903116875215568913679808238264	13.467042454042638170007261022024409	13.470155360061132928954773338423225
13.473213802552343969176414720413487	13.476302647287223508754966768268084	13.479337389227819997111748990692973
13.482346421441562120101495301037430	13.485293318045282335887952362450478	13.488249148435842618350149753783486
13.491179584354353752611550171676095	13.494084747058831414056697557041621	13.497011084300140602722763616549800
13.499895368095523512785232116864023	13.502701722883121030342927584362952	13.505547791585498408392624616902183
13.508333602964302346321586079889387	13.511107842566379075799459233305947	13.513936036377083173562688008219201
13.516654351365134683610700727371316	13.519431847277609941232036080607544	13.522232264247358404187723455476999
13.525000203908312007517604856476096	13.527641315577516333811095751023520	13.530300873952004160088528545275672
13.532917871456062697723758553984577	13.535493461803957879581733368101521	13.538046382601181802194312557622836
13.540554273647847830847507251037425	13.543084323366200838300039176315332	13.545599058351387123157120673148797
13.548063007209007082860967768914526	13.550504412576211288095020257342519	13.552923260697021481475229482726246
13.555319528240038802193441194513629	13.557762668158072770433170818766971	13.560302341521164494357743132135167
13.562641374331046546303564833879801	13.564958044304722734009028672257896	13.567252292374466826596567631465810
13.569777875695742862069575716786180	13.572043884450805383578987603438786	13.574288148599622255870938353612831
13.576510610487748479571836167835301	13.578837748814414710933467280925295	13.581263278311749328511560181131020
13.584419560999735066923148194498099	13.586612599432547359359203779080153	13.588785775454742632035040598882148
13.590831265318841752277570171711095	13.592959911744010496809257658001969	13.594915982519419341022644469998104
13.596998211959713327539754998388164	13.599059952154497874738089781219857	

11. ACKNOWLEDGEMENTS

We wish to thank Andrew Rechnitzer for many stimulating discussions on this topic, and Murray Elder for helpful comments on the manuscript. AJG wishes to thank Nathan Clisby for his vastly superior version of the program to use differential approximants to predict further terms and ratios. AEP wishes to thank ACEMS for financial support through a PhD top-up scholarship.

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