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Analysis of series expansions for non-algebraic singularities

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Abstract

An underlying assumption of existing methods of series analysis is that singularities are algebraic. Functions with such singularities have their n th coefficient behaving asymptotically as $A \cdot \mu^n \cdot n^g$. Recently, a number of problems in statistical mechanics and combinatorics has been encountered in which the coefficients behave asymptotically as $B \cdot \mu^n \cdot \mu_1^{n^\sigma} \cdot n^g$, where typically σ is a simple, rational number between 0 and 1. Identifying this behaviour, and then extracting estimates for the critical parameters B , μ , μ_1 , σ , and g presents a significant numerical challenge. We describe methods developed to meet this challenge.

Keywords: series expansions, singularity analysis, Dyck paths, non-algebraic singularities, interacting SAWs

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(Some figures may appear in colour only in the online journal)

1. Introduction

The method of series analysis has, for many years, been a powerful tool in the study of many problems in statistical mechanics, combinatorics, fluid mechanics and computer science. In essence, the problem is the following: given the first N coefficients of the series expansion of some function, (where N is typically as low as 5 or 6, or as high as 100 000 or more), determine the asymptotic form of the coefficients, subject to some underlying assumption about the asymptotic form, or, equivalently, the nature of the singularity of the function.

Typical examples include the susceptibility of the Ising model, and the generating function of self-avoiding walks (SAWs). These are believed to behave as

$$F(z) = \sum_n c_n z^n \sim C \cdot (1 - z/z_c)^{-\gamma}. \quad (1)$$

In the Ising case, for regular two-dimensional (2D) lattices, the values of both z_c and $\gamma = 7/4$ are exactly known, and the amplitude C is known to more than 100 decimal places. In the SAW case, the value of z_c is only known for the hexagonal lattice [10], and the value of $\gamma = 43/32$ is universally believed, but not proved.

The method of series analysis is used when one or more of the critical parameters is not known. For example, for the three-dimensional versions of the above problems, none of the quantities C , z_c or γ are exactly known. From the binomial theorem it follows from (1) that

$$c_n \sim \frac{C}{\Gamma(\gamma)} \cdot z_c^{-n} \cdot n^{\gamma-1}. \quad (2)$$

Here C , z_c , and γ are referred to as the critical amplitude, the critical point and the critical exponent respectively.

The aim of series analysis is to obtain, as accurately as possible, estimates of the critical parameters from the first N coefficients. Since obtaining these coefficients is typically a problem of exponential complexity, the usual consequence is that fewer than 100 terms are known (and in some cases far fewer)¹.

There are literally thousands of such problems in statistical mechanics, combinatorics, computer science and fluid mechanics (and other areas) where such a situation arises.

The methods to extract estimates of the critical parameters from the known series expansion largely fall into two classes. One class is based on the *ratio method*, initially developed by Domb and Sykes [8], and subsequently refined and expanded by many authors.

The second is based on analysing a differential equation the solution of which has an algebraic singularity (1). It is constructed so that the first N terms of the power series expansion of its solution precisely agree with the known expansion coefficients of the underlying problem. The first development of this nature was due to Baker [1], based on taking Padé approximants of the logarithmic derivative of known series. This was then substantially extended by Guttmann and Joyce [19] who developed the *method of differential approximants* (DAs), which is still the most successful method in use today for analysing series with algebraic singularities, typified by (1).

While, as noted, many problems have such an algebraic singularity structure, an increasing number of situations has been encountered in which a more complex singularity structure prevails. Those cases are characterized by coefficients with dominant asymptotics of the form

$$b_n \sim C \cdot \mu^n \cdot \mu_1^{n^\sigma} \cdot n^g. \quad (3)$$

The term μ^n , where $\mu = 1/z_c$ is the dominant term, then there is a sub-dominant term $\mu_1^{n^\sigma}$, giving rise to two additional parameters, μ_1 and σ . If $\mu_1 > 1$, we can write down a generic generating function whose coefficients have this asymptotic behaviour, but if $\mu_1 < 1$, a generic generating function does not appear to be known, (at least not by the author).

The purpose of this paper is to develop numerical techniques for the analysis of problems whose coefficients have the asymptotic form (3). These techniques will be seen to be extensions of the classical ratio method and DAs method used in the analysis of functions with singularities of the more common asymptotic form (2).

For all the problems we have so far encountered, the exponent σ appearing in equation (3) is a simple rational fraction, and for the problems we discuss here, it turns out to

¹ In the case of the susceptibility of the 2D Ising model, polynomial time algorithms for enumerating the coefficients have been developed [6, 27], and in that case we have hundreds of terms. Unfortunately, this is a rare situation.

lie in the range $[0, \frac{1}{2}]$, though this is by no means universal. For the interacting partially directed self-avoiding walk (IPDSAW) model, defined in section 8, and more general models of the collapse transition in interacting walk models [22], there are compelling physical arguments based on the presence of a surface free-energy that can be shown [11] to give rise to such a term with exponent $\sigma = (d - 1)/d$, where d is the dimensionality of the system². In other cases, such as Dyck paths counted by both length and height, as discussed in section 7, probabilistic arguments³, based on the expected behaviour in the scaling limit of the objects being counted, can be used to prove that $\sigma = 1/3$. For 2D SAWs and bridges, subject to a compressive force at their highest vertex, we have recently shown (non-rigorously) [2] that $\sigma = \frac{3}{7}$.

While the existence of such asymptotic behaviour has been proved in some cases, as mentioned above, in other cases, such as that of compressed SAWs and bridges [2], this behaviour is merely conjectural. Another known case is given by the coefficients of the exponential generating function (EGF) of fragmented partitions [15]. In other cases one has field theoretical arguments, such as those used by Duplantier and colleagues [11–13] in their discussion of 2D collapsed dense polymers, and multiple Manhattan lattice walks, and careful numerical work based on series expansions of an exact solution [3] of IPDSAW by Owczarek *et al* [26].

Another important observation is that for functions whose coefficients have the asymptotic form (2), the underlying generating function has, almost invariably, an algebraic singularity of the form (1). For functions whose coefficients have the asymptotic form (3), the underlying generating function can be a well-behaved D -finite function (as in the case of fragmented permutations), or a function with a natural boundary (as in the case of integer partitions), as well as perhaps something in between of which we don't have an example. So while for algebraic singularities one can perhaps carelessly fail to distinguish between the singularity and its asymptotic form, one must be much more careful when discussing series whose coefficients behave like (3). For want of a better name, we'll refer to these as non-algebraic singularities, while accepting that that describes a much wider class of singularities than those considered here. In this work we develop methods to identify the asymptotic form of the coefficients, assuming it is (2) or (3). We will have nothing to say about the underlying singularity.

Very recently, my colleagues and I have come across several situations in which this generic asymptotic behaviour seems to arise. In combinatorics the notoriously unsolved problem of 1324 pattern-avoiding permutations has, conjecturally, this asymptotic behaviour [7]. A number of 2D SAW problems in which the walk is subject to a compressive force also have coefficients with this asymptotic form [2]. These include self-avoiding bridges, SAWs and polygons. In these models we consider the situation in which the bridge/walk/polygon originates in a horizontal line. The two-variable generating function is

$$G(x, y) = \sum_{n,h} g_{n,h} x^n z^h,$$

where $g_{n,h}$ is the number of objects of length n with maximal height (y -coordinate) h . If $z < 1$, then squat, broad objects are favoured over tall, slim objects. This then models a compressive force applied to the object. For these models, in the compressed regime, there are physical arguments (as alluded to above) that make plausible the existence of the asymptotic structure (3), and these models will be discussed in future publications, currently in preparation⁴.

² Clearly, this permits $\sigma > 1/2$ for $d > 2$.

³ R Pemantle, private communication.

⁴ A number of papers by various subsets of N Beaton, A J Guttmann, I Jensen, E J Janse van Rensburg and S G Whittington are currently being written.

Given the increasingly frequent occurrence of problems where coefficients have such an asymptotic form, it has become pressing to develop numerical techniques to estimate the various critical parameters. That is the purpose of this article. We first outline the two principal methods used to analyse algebraic singularities. We then discuss how these methods behave when applied to the class of non-algebraic singularity we are considering here. Naturally, they fail in these cases, but the nature of their failure gives information about the true nature of the singularity.

We then show how the ratio method can be modified and extended to be useful in analysing these non-algebraic singularities, and how the method of DAs can also be applied to provide useful information. In the next section we describe the traditional ratio method. In the following two sections we describe the method of Padé approximants, and then the method of DAs, showing just how precise estimates can be obtained in favourable circumstances. Our main purpose here is to show just how good the DA method is in estimating the critical parameters of algebraic singularities. By contrast, it performs very poorly when given a series possessing the type of non-algebraic singularity considered here. It is precisely this poor performance that indicates the presence of this type of singularity⁵.

In section 5 we discuss this particular non-algebraic asymptotic behaviour, and give a generic OGF that has coefficients with the appropriate asymptotic form (when $\mu_1 \geq 1$). We show how to modify and extend the ratio method so that it can be used in such situations. We then discuss the application of the method of DAs to such non-physical singularities, and show how the coefficients can be transformed to new coefficients which behave, to leading order, like those of an algebraic singularity.

In subsequent sections we study three examples, of increasing difficulty, which are known to have non-algebraic singularities with the asymptotic behaviour (3) considered here. Our first example is a slightly modified version of the generating function for fragmented permutations. The second is an analysis of Dyck paths subjected to a compressing force at their top vertex, and our final example is that of IPDSAWs.

We then conclude by giving a method, or more precisely a number of methods, which collectively provide an effective recipe for analysing series expansions with coefficients of this non-algebraic asymptotic form. Furthermore, the methods provide effective tools for *predicting* that the asymptotic form is of the presumed type.

2. Ratio method

The ratio method was perhaps the earliest systematic method of series analysis employed, and is still a useful starting point, prior to the application of more sophisticated methods. It was first used by M F Sykes in his 1951 D Phil studies, under the supervision of C Domb. From equation (2), it follows that the ratio of successive terms

$$r_n = \frac{c_n}{c_{n-1}} = \frac{1}{z_c} \left(1 + \frac{\gamma - 1}{n} + o\left(\frac{1}{n}\right) \right). \quad (4)$$

From this result, it is then natural to plot the successive ratios r_n against $1/n$. If the correction terms $o(\frac{1}{n})$ can be ignored⁶, such a plot will be linear, with gradient $\frac{\gamma-1}{z_c}$, and intercept $1/z_c$ at $1/n = 0$.

⁵ Indeed, this observation was the catalyst for this work. Our series analysis in a problem of compressed polygons we were studying behaved so uncharacteristically badly, we were driven to find out why, and this work is the result.

⁶ For a purely algebraic singularity (1), with no confluent terms, the correction term will be $O(\frac{1}{n^2})$.

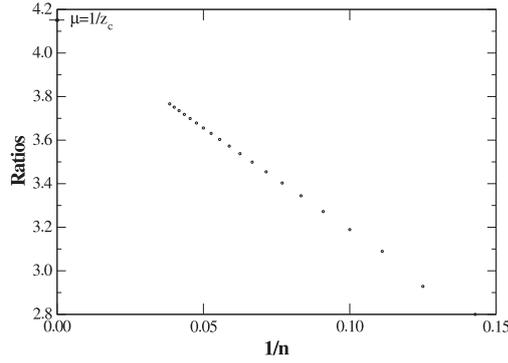


Figure 1. Plot of ratios against $1/n$ for triangular lattice polygons. A straight line through the last few data points intercepts the ratios axis at $1/z_c$.

As an example, we apply the ratio method to the generating function of self-avoiding polygons (SAPs) on the triangular lattice. The first few terms in the generating function, from p_3 to p_{26} are: 2, 3, 6, 15, 42, 123, 380, 1212, 3966, 13 265, 45 144, 155 955, 545 690, 1930 635, 6897 210, 24 852 576, 90 237 582, 329 896 569, 1 213 528 736, 4 489 041 219, 16 690 581 534, 62 346 895 571, 233 893 503 330, 880 918 093 866. Plotting successive ratios against $1/n$ results in the plot shown in figure 1. The critical point is estimated [23] to be at $z_c \approx 0.240\,917\,574\dots = 1/4.150\,797\,22\dots$. From the figure one sees that the locus of points, after some initial (low n) curvature, becomes linear to the naked eye for $n > 15$ or so, (corresponding to $1/n < 0.067$). Visual extrapolation to $1/z_c$ is quite obvious. A straight line drawn through the last 4 – 6 data points intercepts the horizontal axis around $1/n \approx 0.13$. Thus the gradient is approximately $\frac{4.1508 - 2.8}{-0.13} \approx -10.39$, from which we conclude that the exponent $\gamma - 1 \approx -10.39 \cdot z_c \approx -2.50$. It is believed [25] that the exact value is $\gamma = -3/2$, which is in complete agreement with this simple graphical analysis.

Various refinements of the method can be readily derived. If the critical point is known exactly, it follows from equation (4) that estimators of the exponent γ are given by

$$\gamma_n = n(z_c \cdot r_n - 1) + 1 = \gamma + o(1).$$

If z_c is unknown, estimates of the exponent γ can be obtained by defining estimators γ_n of γ and extrapolating these against $1/n$. Here

$$\gamma_n = 1 + n^2 \left(1 - \frac{r_n}{r_{n-1}} \right) = \gamma + o(1). \tag{5}$$

Similarly, if the exponent γ is known, estimators of the critical point z_c are given by

$$z_c^{(n)} = \frac{n + \gamma - 1}{nr_n} = z_c + o(1).$$

One problem with the ratio method is that if the singularity closest to the origin is not the singularity of interest (the so-called *physical singularity*), then the ratio method will not give information about the physical singularity. Worse still, if the closest singularity to the origin is a conjugate pair of singularities, the ratios will vary dramatically in both sign and magnitude. We see precisely this behaviour in the low-temperature series of the three-dimensional Ising model [9]. There, we see for the simple-cubic lattice the dominant singularity is on the negative real axis, while for the body-centred and face-centred cubic lattice series, the closest singularities to the origin are one or more conjugate pairs. To overcome this difficulty, and in

effect apply an analytic continuation, Baker [1] proposed the use of Padé approximants applied to the logarithmic derivative of the series expansion.

We should also mention that there exists a vast literature of extrapolation techniques in numerical analysis, and many such methods can be advantageously applied to extrapolate the sequence of ratios in order to estimate the radius of convergence, which is the critical point. Some of these methods, applied to series analysis problems, are discussed in the review [17]. In particular, the Bulirsch–Stoer algorithm [5] has been found to be quite powerful, as it allows for the more general situation when convergence is not linear in $1/n$. (Recall that, for an isolated algebraic singularity, convergence is always linear in $1/n$, so in that case the Bulirsch–Stoer method affords no advantage.) We have used the Bulirsch–Stoer method to estimate the radius of convergence in all the examples with non-algebraic singularities that we consider below. A short, annotated Maple code both defining and implementing the algorithm is given as an appendix.

3. Padé approximants

The basic idea of using Padé approximants for series analysis is very simple. Given a function $F(z)$ with a simple pole at some point z_c we use the series expansion of $F(z)$ to form a rational approximation to $F(z)$,

$$F(z) = \frac{P_i(z)}{Q_j(z)}, \quad (6)$$

where $P_i(z)$ and $Q_j(z)$ are polynomials of degree i and j respectively, whose coefficients are chosen such that the first $i + j + 1$ terms in the series expansion of $F(z)$ are identical to those of the expansion of $P_i(z)/Q_j(z)$, with $Q_j(0) = 1$ for uniqueness. Constructing the polynomials only involves solving a system of linear equations.

In order to use the Padé approximation scheme to reliably approximate an algebraic singularity rather than just a meromorphic functions, we must first transform the series into a suitable form. This brings us to the classic method called Dlog-Padé approximation [1]. If we have a function with expected behaviour typical of algebraic singular points, as given by equation (1), then taking the derivative of the logarithm of $F(z)$ gives

$$\widehat{F}(z) = \frac{d}{dz} \log F(z) \simeq \frac{\gamma}{z_c - z} + O(1). \quad (7)$$

This form is perfectly suited for Padé analysis, as taking the logarithmic derivative has turned the function into a meromorphic function (at least to leading order). We see that an estimate of the critical point z_c can be obtained from the roots of the denominator polynomial $Q_j(z)$, while an estimate of the critical exponent γ is obtainable from the residue of the Padé approximant to $\widehat{F}(z)$ at z_c , that is

$$\gamma \approx \lim_{z \rightarrow z_c} (z_c - z) \frac{P_i(z)}{Q_j(z)}. \quad (8)$$

As is the case for the ratio method, refinements exist for those situations when the critical point or critical exponent is exactly known [17].

Since $\widehat{F}(z) = F'(z)/F(z)$, we see that forming a Dlog-Padé approximant is simply equivalent to seeking an approximation to $F(z)$ by solving the first order linear homogeneous differential equation, with polynomial coefficients

$$F'(z)Q_j(z) - F(z)P_j(z) = 0.$$

This observation leads us directly to the more powerful and more general method of DAs by noting that we can approximate $F(z)$ by a solution to a higher order linear ODE (possibly inhomogeneous), with polynomial coefficients. This method was first proposed and developed by Guttmann and Joyce [19] in 1972, and was subsequently extended to the inhomogeneous case by Au-Yang and Fisher [14] and Hunter and Baker [20] in 1979. The advantage of a higher order ODE is that confluent singularities can be accommodated [28], as well as a more complicated singularity structure in general. Many problems in critical phenomena (for example SAWs and the Ising model in three dimensions) have singularity structures of the form

$$F(z) \sim A \cdot (1 - z/z_c) \left[1 + (1 - z/z_c)^\Delta \right],$$

where the so-called *confluent exponent* $0 < \Delta < 1$. The case $\Delta = 1$ corresponds to the absence of a confluent term. Functions that satisfy such an ODE (linear, with polynomial coefficients) are called *D-finite* or *holonomic*.

4. DAs

The generating functions of many lattice models in statistical mechanics and combinatorics are often algebraic, or otherwise given by the solution of simple linear ODEs. This observation (originally made in the context of the 2D Ising model) is the origin of the method of DAs. The basic idea is to approximate a generating function $F(x)$ by solutions of differential equations with polynomial coefficients. The singular behaviour of such ODEs is a well known classical mathematics problem (see e.g. [16, 21]) and the singular points and exponents are easily calculated. Even if *globally* the function is not describable by a solution of such a linear ODE (as is frequently the case) one hopes that *locally*, in the vicinity of the (physical) critical points, the generating function is still well-approximated by a solution to a linear ODE.

An M th-order DA to a function $F(z)$ is formed by matching the coefficients in the polynomials $Q_k(z)$ and $P(z)$ of degree N_k and L , respectively, so that the formal solution of the M th-order inhomogeneous ordinary differential equation

$$\sum_{k=0}^M Q_k(z) \left(z \frac{d}{dz} \right)^k \tilde{F}(z) = P(z) \tag{9}$$

agrees with the first $N = L + \sum_k (N_k + 1)$ series coefficients of $F(z)$. Constructing such ODEs only involves solving systems of linear equations. The function $\tilde{F}(z)$ thus agrees with the power series expansion of the (generally unknown) function $F(z)$ up to the first N series expansion coefficients. We normalize the DA by setting $Q_M(0) = 1$, thus leaving us with N rather than $N + 1$ unknown coefficients to find, in order to specify the ODE. The choice of the differential operator $z \frac{d}{dz}$ in (9) forces the origin to be a regular singular point. The reason for this choice is that most lattice models with holonomic solutions, for example, the free-energy of the 2D Ising model, possess this property. However this is not an essential choice.

From the theory of ODEs, the singularities of $\tilde{F}(z)$ are approximated by zeros z_i , $i = 1, \dots, N_M$ of $Q_M(z)$, and the associated critical exponents γ_i are estimated from the indicial equation. If there is only a single root at z_i this is just

Table 1. Critical point and exponent estimates for self-avoiding polygons. Numbers in parentheses give the uncertainty in the last quoted digits.

L	Second order DA		Third order DA	
	x_c^2	$2 - \alpha$	x_c^2	$2 - \alpha$
0	0.292 893 218 54(19)	1.500 000 65(41)	0.292 893 218 65(12)	1.500 000 40(28)
5	0.292 893 218 75(21)	1.500 000 10(59)	0.292 893 218 52(48)	1.500 000 41(99)
10	0.292 893 218 55(23)	1.500 000 60(48)	0.292 893 218 78(32)	1.499 999 99(97)
15	0.292 893 218 59(19)	1.500 000 54(43)	0.292 893 218 61(37)	1.500 000 35(67)
20	0.292 893 218 66(15)	1.500 000 38(33)	0.292 893 218 60(21)	1.500 000 49(43)

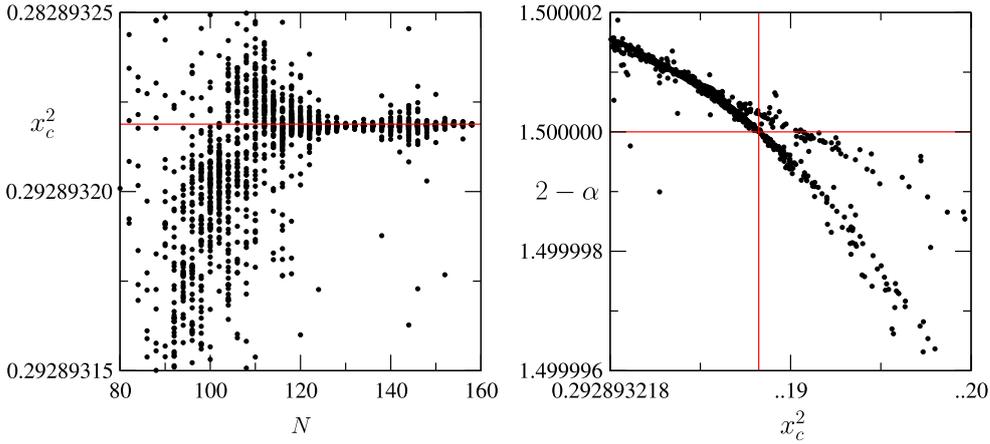


Figure 2. Plot of estimates from third order differential approximants for x_c^2 versus the highest order term used, and the right panel shows $2 - \alpha$ versus x_c^2 . The straight lines are the exact predictions.

$$\gamma_i = M - 1 - \frac{Q_{M-1}(z_i)}{z_i Q'_M(z_i)}. \tag{10}$$

Details as to which approximants should be used and how the estimates from many approximants are combined to give a single estimate are given in [18]. In the next sub-section we give an example of the application of the method.

4.1. The honeycomb SAP generating function

In this sub-section we apply the method of DAs to the generating function for SAPs on the honeycomb lattice. The generating function

$$P(x) = \sum_{n \geq 1} p_n x^{2n}$$

is expected to have a dominant singularity $const. \cdot (1 - x^2/x_c^2)^{2-\alpha}$. On this lattice the critical point is known rigorously [10], and the critical exponent and some universal amplitude ratios are believed to be known exactly. In table 1 we have listed the estimates for the critical point x_c^2 and exponent $2 - \alpha$ obtained from second- and third-order DAs. We note that all the

estimates are in agreement in that within ‘error-bars’ they take the same value. From this we arrive at the estimate $x_c^2 = 0.292\,893\,2186 \pm 5 \times 10^{-10}$ and $2 - \alpha = 1.500\,0004 \pm 1 \times 10^{-6}$. The final estimates are in perfect agreement with the exact values [10] $x_c^2 = 1/\mu^2 = 1/(2 + \sqrt{2}) = 0.292\,893\,218\,813\dots$ and $2 - \alpha = 3/2$.

Not surprisingly, the estimates improve as the number of available series terms increases. This can be seen in the left panel of figure 2 where the estimates x_c^2 obtained from third-order DAs are plotted against the highest order coefficient $N < N_{\max}$ used in constructing the DA. Each dot in the figure is an estimate obtained from a specific approximant. As can be seen, the estimates clearly settle down to the conjectured exact value (solid line) as N is increased, and there is no evidence of any systematic drift at large N .

In the right-hand panel we show the variation in the exponent estimates with the critical point estimates. Thus if one knows or conjectures either the exponent or critical point, a more precise estimate of the other can be obtained. The ‘curve’ traced out by the estimates passes through the intersection of the lines given by the exact values. The apparent branching into two arcs is probably spurious.

One of the reasons for giving this example is to show just how successful and precise the method is under favourable circumstances. We argue that, by contrast, when the method behaves badly, with poorly converged estimates of the radius of convergence and wildly varying exponent estimates, that this is a signal that the underlying singularity is not an algebraic singularity. In the next section we discuss other types of singularities that give rise to a more complicated asymptotic form.

5. Functions with non-algebraic singularities

A number of solved, and, we claim, unsolved problems that arise in lattice critical phenomena and algebraic combinatorics have coefficients with a more complex asymptotic form, with a sub-dominant term $O(\mu_1^{n^\sigma})$ rather than $O(n^g)$. In fact the sub-sub dominant term is $O(n^g)$. Perhaps the best-known example of this sort of behaviour is the number of partitions of the integers—though in that case the leading exponential growth term μ^n is absent (or equivalently $\mu = 1$).

There are a number of models in mathematical physics that also have this more complex asymptotic structure. For example, Duplantier and Saleur [13] and Duplantier and David [12] studied the case of *dense* polymers in two dimensions, and found the partition functions had the asymptotic form

$$Q_n \sim \text{const.} \cdot \mu^n \cdot \mu_1^{n^\sigma} \cdot n^g.$$

In [26], Owczarek *et al* investigated an exactly solvable model of IPDSAW, for which the solution had previously been given by Brak *et al* in [3]. In particular they analysed a 6000 term series expansion for IPDSAWs in the collapse regime, and estimated $\sigma = 1/2$, $g = -3/4$, while μ_1 was found to at least six digit accuracy. From [3] the value of μ is exactly known. Subsequently Duplantier [11] pointed out that $\sigma = 1/2$ is to be expected, not only for IPDSAWs, but also for SAWs in the collapsed regime. In section 8 we show that the methods we develop below can give good results using only about 100 terms (occasionally 200), rather than 6000 used in [28]. This is of practical importance, as for many unsolved problems one typically only has 20–200 terms available.

An example from combinatorics is given by the EGF of fragmented permutations⁷ [15] which is

$$F(z) = \exp\left(\frac{z}{1-z}\right).$$

Then with $F_n = [z^n]F(z)$, we have [15], p563

$$F_n \sim \frac{e^{2\sqrt{n}}}{2\sqrt{\pi e} \cdot n^{3/4}}.$$

This follows from Wright [29, 30] who calculated the leading asymptotic form of the expansion of

$$F(z) = (1 - \mu z)^{-\beta} \exp\left(\frac{A}{(1 - \mu z)^\rho}\right), \quad A > 0, \quad \rho > 0. \quad (11)$$

For $\rho \leq 1$, Wright's saddle-point analysis yields

$$[z^n]F(z) \sim \mu^n \frac{N^{\beta-1-\rho/2} \exp(A(\rho+1)N^\rho)}{\sqrt{2\pi A \rho (\rho+1)}}, \quad (12)$$

with $N := \left(\frac{n}{A\rho}\right)^{\frac{1}{\rho+1}}$.

This asymptotic form can be written as

$$B \cdot \mu^n \cdot \mu_1^{n^\sigma} \cdot n^g, \quad (13)$$

where $\mu_1 = \exp((\rho+1) \cdot A^{\frac{\rho}{\rho+1}} \cdot \rho^{-\frac{1}{\rho+1}})$, so in particular $\mu_1 > 1$. Also $\sigma = \frac{\rho}{\rho+1}$, and $g = \frac{\beta-1-\rho/2}{\rho+1}$.

For this situation, with $\mu_1 > 1$, the term involving $\mu_1^{n^\sigma}$ rapidly dominates the term n^g , for any value of g . However if $\mu_1 < 1$, the term $\mu_1^{n^\sigma}$ is eventually smaller than the contribution of the term n^g . For the situation $\mu_1 < 1$, we are unaware of any analogue of Wright's expansion. That is to say, we do not know what generic closed form expression, analogous to (11), has an asymptotic expansion of the form (13) with $\mu_1 < 1$.

In the remainder of this paper we develop numerical methods to analyse functions whose coefficients have the asymptotic form given in equation (13), based on extensions of the ratio method and the method of DAs. We then take three examples of functions of increasing complexity with coefficients that are known to behave asymptotically as in equation (13) and see how successful or otherwise the methods are.

5.1. Ratio method for non-algebraic singularities

If the coefficients of some generating function behave as

$$b_n \sim B \cdot \mu^n \cdot \mu_1^{n^\sigma} \cdot n^g, \quad (14)$$

⁷ A fragmented permutation is an unordered collection of non-empty sub-permutations of a given permutation. For example, there are three fragmented permutations of two elements: {1, 2}, {2, 1} and {1}, {2}.

then the ratio of successive coefficients $r_n = b_n/b_{n-1}$, is

$$r_n = \mu \left(1 + \frac{\sigma \log \mu_1}{n^{1-\sigma}} + \frac{g}{n} + \frac{\sigma^2 \log^2 \mu_1}{2n^{2-2\sigma}} + \frac{(\sigma - \sigma^2) \log \mu_1 + 2g\sigma \log \mu_1}{2n^{2-\sigma}} + \frac{\sigma^3 \log^3 \mu_1}{6n^{3-3\sigma}} + O(n^{2\sigma-3}) + O(n^{-2}) \right). \tag{15}$$

In the examples considered in this paper, σ takes the simple values 1/2 or 1/3. When $\sigma = \frac{1}{2}$ (15), specializes to

$$r_n = \mu \left(1 + \frac{\log \mu_1}{2\sqrt{n}} + \frac{g + \frac{1}{8} \log^2 \mu_1}{n} + \frac{\log^3 \mu_1 + (6 + 24g) \log \mu_1}{48n^{3/2}} + O(n^{-2}) \right), \tag{16}$$

and when $\sigma = \frac{1}{3}$, to

$$r_n = \mu \left(1 + \frac{\log \mu_1}{3n^{2/3}} + \frac{g}{n} + \frac{\log^2 \mu_1}{18n^{4/3}} + \frac{(2 + 6g) \log \mu_1}{18n^{5/3}} + O(n^{-2}) \right). \tag{17}$$

So, given a series, if one applies the ratio method and finds the ratio plots are not linear, and can be linearized by plotting the ratios against $1/n^{1-\sigma}$, with $\sigma = 1/2$ or $1/3$, then this suggests that the asymptotic form of the coefficients could well be of the type considered here. Unfortunately this observation does not provide a sufficiently precise method to estimate the value of σ . One can perhaps distinguish between $\sigma = 1/2$ and $\sigma = 1/3$ in this way, but one cannot be much more precise than that. However, as we now show, one can extend the ratio method to provide estimates for the value of σ .

From (15), one sees that

$$(r_n/\mu - 1) = \sigma \log \mu_1 \cdot n^{\sigma-1} + O\left(\frac{1}{n}\right). \tag{18}$$

Accordingly, a plot of $\log (r_n/\mu - 1)$ versus $\log n$ should be linear, with gradient $\sigma - 1$. We would expect an estimate of σ close to that which linearized the ratio plot.

A second estimator of σ can be constructed from (13) by observing that

$$c_n = \log \left(\frac{b_n}{\mu^n} \right) \sim \log B + \log \mu_1 \cdot n^\sigma + g \log n, \tag{19}$$

so that

$$\log c_n = \text{const.} + \sigma \log n + O(n^{-\sigma} \log n). \tag{20}$$

So a plot of $\log (c_n)$ against $\log n$ should be linear, with gradient σ . Both estimators will usually provide visually linear plots, but the local gradients are changing as n increases. It is therefore worthwhile extrapolating the local gradients. To do this, from (18), we form the estimators

$$\tilde{\sigma}_n = 1 + \frac{\log |r_n/\mu - 1| - \log |r_{n-1}/\mu - 1|}{\log n - \log (n - 1)}. \tag{21}$$

This can be extrapolated against $1/n^\sigma$, using any approximate value of σ . The local gradient of the second estimator, given by equation (20), can be similarly extrapolated, and similarly plotted against $1/n^\sigma$, using any approximate value of σ .

Both these ways of estimating σ requires knowledge of, or at worst a very precise estimate of, the growth constant μ . While μ is exactly known in the three examples considered below, more generally μ is not known, and must be estimated, along with all the other critical parameters. In order to estimate σ without knowing μ , we can use one (or both) of the following estimators:

From equation (15), it follows that

$$r_{\sigma_n} = \frac{r_n}{r_{n-1}} \sim 1 + \frac{(\sigma - 1) \log \mu_1}{n^{2-\sigma}} + O\left(1/n^2\right), \tag{22}$$

so σ can be estimated from a plot of $\log(r_{\sigma_n})$ against $\log n$, which should have gradient $\sigma - 2$. Again, the local gradients can be estimated and plotted against $1/n^\sigma$, using any approximate value of σ .

Another estimator of σ when μ is not known follows from equation (14),

$$a_{\sigma_n} = \frac{b_n^{1/n}}{b_{n-1}^{1/(n-1)}} \sim 1 + \frac{(\sigma - 1) \log \mu_1}{n^{2-\sigma}} + O\left(1/n^2\right), \tag{23}$$

so again σ can be estimated from a plot of $\log(a_{\sigma_n})$ against $\log n$. Again, estimates of σ can be improved by extrapolating the local gradient against $1/n^\sigma$.

While these two estimators are equal to leading order, they differ in their higher-order terms. And indeed, as shown below, which of the two is more informative varies from problem to problem.

5.2. Direct fitting for non-algebraic singularities

Another, perhaps obvious, idea is to try and fit the critical parameters directly to the assumed asymptotic form

$$b_n \sim B \cdot \mu^n \cdot \mu_1^{n^\sigma} \cdot n^g.$$

Therefore

$$\log b_n \sim \log B + n \log \mu + n^\sigma \log \mu_1 + g \log n. \tag{24}$$

So if σ is known, or assumed, there are four unknowns in this linear equation. It is then straightforward to solve the linear system

$$\log b_k = c_1 k + c_2 k^\sigma + c_3 \log k + c_4$$

for $k = n - 2, n - 1, n, n + 1$ with n ranging from 3 to $N - 1$, where N is the power of the highest known series coefficient. Then c_1 estimates $\log(\mu)$, c_2 estimates $\log(\mu_1)$, c_3 estimates g and c_4 gives estimators of $\log B$. An obvious variation arises in those cases where, say, μ is known. Then one can solve

$$b_k - k \log \mu = c_2 k^\sigma + c_3 \log k + c_4$$

from three successive coefficients, as before increasing the order of the lowest used coefficient by one until one runs out of coefficients.

5.3. Using the method of DAs

In this sub-section we investigate the use of the method of DAs in the analysis of series with asymptotic coefficients of the form (3). We will see in our first example—a modified version of the generating function for fragmented permutations—that the EGF is in fact holonomic, satisfying a first-order linear ODE. So an appropriately chosen DA will solve this problem

completely, based on only a few terms in the series expansion. So this is not a testing example.

Our second example, that of height-weighted Dyck paths, is more typical. If one simply applies the method of DAs, the results, discussed in section 7, suggest that the generating function is not well-approximated by a linear ODE of the assumed type—and hence that the singularity is not likely to be algebraic. This behaviour is typical of those cases where the singularity is not of the assumed algebraic type. That is to say, in such cases one typically sees imprecise and inaccurate estimates of the critical point, unrealistic values of the associated critical exponent, and sometimes a concentration of other critical points along the real axis. This behaviour is characteristic of the situation in which the DAs are trying unsuccessfully to represent the singularity(ies) of the underlying generating function.

Earlier in this section we discussed Wright’s function (11), which generates coefficients of the asymptotic form (3) considered here. However, as discussed, Wright’s function only generates asymptotic forms for its coefficients when $\mu_1 > 1$. For those situations when $\mu_1 < 1$, Wright’s function does not generate coefficients of the required asymptotic form. Indeed, if one asks the natural question, ‘what OGF has coefficients with asymptotic behaviour $\text{const.} \cdot 4^n \cdot \mu_1^{n^{1/3}} \cdot n^g$, where $\mu_1 < 1$?’ the answer seems to be unknown⁸.

We can (partially) side-step this difficulty by constructing an OGF with coefficients which are just the reciprocals of the original coefficients. For if $c_n \sim \text{const.} \cdot \mu^n \cdot \mu_1^{n^\sigma} \cdot n^g$, then $d_n = 1/c_n \sim \text{const.} \cdot \mu^{-n} \cdot (1/\mu_1)^{n^\sigma} \cdot n^{-g}$. So if $\mu_1 < 1$ the coefficients are now of a form given by the asymptotic expansion of Wright’s function. Unfortunately, knowing the properties of a series $\sum c_n z^n$ tells us little about the properties of $\sum z^n/c_n$, and our numerical experiments with this transformation have, with isolated exceptions, been largely unsuccessful.

Note that if we take the logarithmic derivative of Wright’s function (11), we obtain

$$\tilde{F}(z) = \frac{d}{dz} \log F(z) = \frac{F'(z)}{F(z)} = \frac{\beta\mu}{1 - \mu z} + \frac{A\rho\mu}{(1 - \mu z)^{\rho+1}}. \tag{25}$$

$\tilde{F}(z)$ now has algebraic singularities, and so might be amenable to analysis by the method of DAs. That is to say, we might expect the logarithmic derivative of the OGF of the reciprocal series to behave as a function with algebraic singularities at $z = 1/\mu$, and with exponents -1 (a simple pole) and a dominant branch point with exponent $-(\rho + 1)$. Numerical experiments show that these transformations—taking the logarithmic derivative of the OGF with reciprocal coefficients—substantially improve the performance of the DAs method for the analysis of non-physical singularities of the assumed type, but, while useful, are not as accurate as we need for a reliable method. Fortunately, we have developed a different transformation that is more effective.

5.4. Transforming series to remove the factor $\mu_1^{n^\sigma}$

As noted above, the method of DAs is of limited use in analysing series which are not dominated by an algebraic singularity. For those series with coefficients with the asymptotic form considered here, it is the presence of the $\mu_1^{n^\sigma}$ term that is responsible for the lack of applicability of the method. However we can manipulate the series to remove the offending term, and then use this powerful method. From equation (24) one has, when $\sigma = 1/2$,

⁸ It is certainly not given by Wright’s OGF with $A < 0$, for in that case the coefficients actually change sign with a known periodicity.

$$\log b_n = \log B + n \log \mu + \sqrt{n} \log \mu_1 + g \log n + O(1/\sqrt{n}).$$

Then with $\tilde{b}_n = \log b_n/\sqrt{n}$, we can form new coefficients c_n :

$$c_n = 2n^{3/2}(\tilde{b}_n - \tilde{b}_{n-1}) = (2g - \log B) + n \log(\mu) - g \log(n) + O(1/\sqrt{n}). \quad (26)$$

Exponentiating these coefficients, we have

$$d_n = \exp(c_n) = D \cdot \mu^n \cdot n^{-g} \cdot 1 + O(1/\sqrt{n}),$$

where $D = e^{2g/B}$.

When $\sigma = 1/3$, one has

$$\log b_n = \log B + n \log \mu + n^{1/3} \log \mu_1 + g \log n + O(1/n^{1/3}).$$

Then defining $\tilde{b}_n = b_n/n^{1/3}$, one has

$$c_n = \frac{3}{2}n^{4/3}(\tilde{b}_n - \tilde{b}_{n-1}) = \frac{3g - \log B}{2} + n \log(\mu) - \frac{g}{2} \log(n) + O(1/n^{1/3}). \quad (27)$$

So in this case

$$d_n = \exp(c_n) = D \cdot \mu^n \cdot n^{-g/2} \cdot 1 + O(1/n^{1/3}),$$

where $D = e^{3g/2/\sqrt{B}}$.

In this way we have transformed the series to one whose coefficients, d_n , behave asymptotically, at least to leading order, like a function with an algebraic singularity. We can therefore analyse the series with transformed coefficients d_n by the method of DAs. Note however that the correction terms are $O(1/n^\sigma)$, whereas for an isolated algebraic singularity they are $O(1/n)$, so one can't expect the standard methods, like the method of DAs, to perform as well with the transformed series as, say, the example in section 4.1.

We can also apply other standard techniques to the analysis of the transformed series. The ratios of successive terms (d_n) of the transformed series when plotted against $1/n$ are now linear, but as the simple ratio method doesn't give us a particularly accurate estimate of μ , we don't give the results here. Rather, we extrapolate the ratios of the coefficients of the transformed series using the Bulirsch–Stoer algorithm, with parameter $w = 1$, as appropriate for an expected correction term $O(1/n)$.

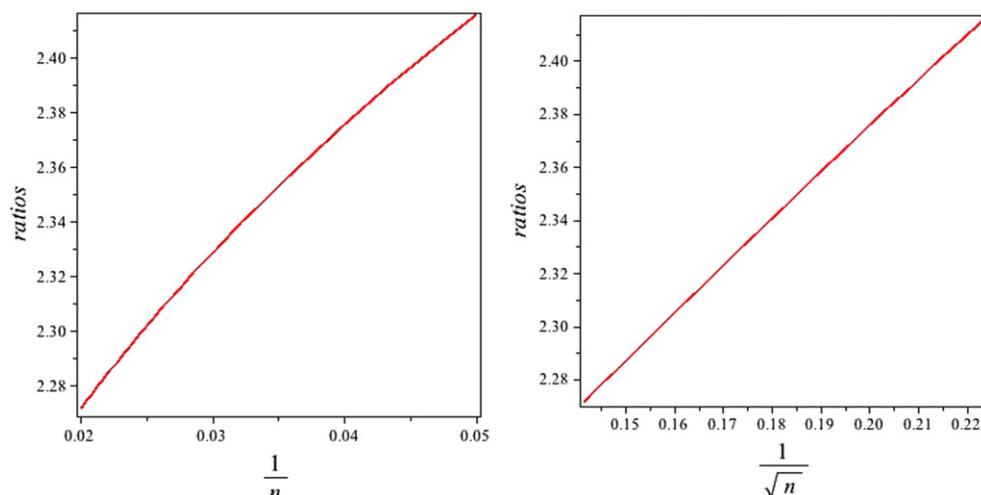
In order to estimate the critical exponent g , we also tried the simple ratio method, extrapolating estimators $g_n = n^2(1 - \frac{r_n}{r_{n-1}})$ of the exponent against $1/n$, as described at equation (5).

In summary, it is clearly useful to transform the original series as described by equations (26), (27) and apply the standard methods of series analysis.

In the next three sections we will consider three problems whose coefficients have the assumed asymptotic form

$$b_n \sim B \cdot \mu^n \cdot \mu_1^{n^\sigma} \cdot n^g.$$

In all cases we first try to estimate the value of σ and μ . After determining the value of σ , the estimate of μ is refined. Next we estimate the other critical parameters μ_1 and g . Finally the amplitude term B is estimated.



(a) Plot of ratios of coefficients of (28) against $\frac{1}{n}$. (b) Plot of ratios of coefficients of (28) against $\frac{1}{\sqrt{n}}$.

Figure 3. Ratio plots for fragmented permutations.

6. Example 1. Modified fragmented partitions

We take as our first example a minor variant of the EGF of fragmented partitions and consider⁹

$$F_1(z) = \exp\left(\frac{2z}{1-2z}\right). \tag{28}$$

Then with $f_n = [z^n]F_1(z)$,

$$f_n \sim 2^{n-1} \frac{e^{2\sqrt{n}}}{\sqrt{\pi e} \cdot n^{3/4}}. \tag{29}$$

$F_1(z)$ is clearly holonomic, satisfying the simple ODE

$$(1-2z)^2 F_1'(z) = 2F_1(z), \quad F_1(0) = 1,$$

but we don't make use of this in the subsequent analysis.

We have generated the series expansion of (28) up to the coefficient of z^{50} to attempt an analysis¹⁰. Applying the ratio method (4) to these coefficients, the resulting plot is shown in figure 3(a). That is to say, we plot the ratios $r_n = \frac{f_n}{f_{n-1}}$ against $\frac{1}{n}$. Unlike the case of an algebraic singularity, with ratio plots shown in figure 1, here one sees considerable curvature in the plot. This is the hallmark of the type of non-algebraic singularity we are considering here.

We show in figure 3(b) the same ratios, but now plotted against $\frac{1}{\sqrt{n}}$. This plot appears to be linear, and also to be approaching the expected limit of 2 as $n \rightarrow \infty$. Plotting the ratios against $\frac{1}{n^{2/3}}$ (not shown) also gives a plot that looks almost as linear as figure 3(b), so trying to

⁹ The introduction of the factor $2z$ rather than z in both numerator and denominator is so that this example will have an exponential growth factor μ of 2, rather than 1 as for fragmented permutations. Our analysis methods seek to identify the dominant exponential growth term.

¹⁰ Of course it is trivial to generate vastly longer series, but a series of 50 terms is not atypical in those frequent cases where the coefficients have to be calculated by some algorithm of exponential complexity.

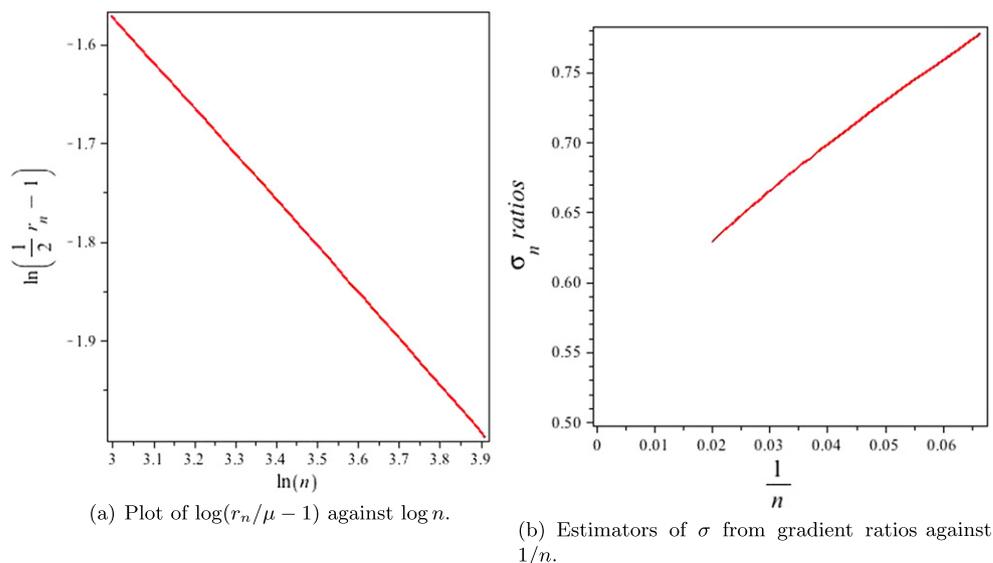


Figure 4. Plots to determine the exponent σ for fragmented permutation asymptotics.

distinguish the correct value of σ in this way is not very precise. The best we can do is to estimate that σ is in the range $[0.4, 0.7]$.

In order to more accurately estimate the value of σ , we show in figure 4(a) a plot of $\log(r_n/\mu - 1)$ versus $\log n$. This is seen to be linear, and the gradient, calculated from the last two ratios, r_{49} and r_{50} is -0.475 . Recall that this gradient should be $\sigma - 1$. If one accepts that σ is a simple rational number, the value of $1/2$ is inescapable. Estimating σ from the second estimator (20) gives a less compelling result, as the gradient, calculated from the last two ratios, gives $\sigma \approx 0.67$. However a plot of these gradients against $1/\sqrt{n}$ is clearly approaching a limiting value around 0.5 .

If we didn't know the value of μ , we could estimate σ from the gradient of plots of $\log(r_{\sigma_n})$ against $\log n$ see equation (22), or $\log(a_{\sigma_n})$, see equation (23). It turns out that they are equally good, and we show in figure 4(b) the estimate of σ given by the gradient of the line joining the points $\log(r_{\sigma_k})$ and $\log(r_{\sigma_{k-1}})$, as k ranges from 15 to 50. There is some curvature in the plot, but clearly a limit of 0.5 is attainable. We extended the plot to 250 terms (not shown), and the curvature increased, making the known limit 0.5 totally evident. A second estimator of σ when μ is not known is given by equation (23). This gives a less accurate estimate of σ , and one that is changing much more rapidly with the order of the coefficients. Again, a plot of these gradients against $1/\sqrt{n}$ is clearly approaching a limiting value around 0.5 .

The point we want to make here is that if one wants to identify σ as a simple fraction, likely to be $1/2$ or $1/3$, then we have good evidence that it is $1/2$. This can then be used in subsequent analysis. We note however that if σ was, say $11/24$ we could be struggling to distinguish this value from $1/2$.

Assuming that $\sigma = 1/2$, we next refine the estimate of μ . We could linearly extrapolate the ratio plot in figure 3(b) which can be seen to be, plausibly, going to a value around 2 on the ordinate, and we might guess that the value was exactly 2. However, more generally μ does not take an integral, nor perhaps even an algebraic, value, so it needs to be estimated

quite precisely. It is therefore necessary to use an extrapolation algorithm which can accommodate the expected asymptotic behaviour of the ratios.

The Bulirsch–Stoer algorithm [5] is such an algorithm, as it extrapolates sequences that behave as $s_n \sim s_\infty + c/n^w$. A Maple implementation is given in the appendix. The parameter w is given by the user. In this example, we set $w = 1/2$, and extrapolate the first 50 ratios. The method produces rows of extrapolants that take into account successively higher powers of order n^{-kw} , with k increasing by 1 with each successive row, as well as terms of order n^{-m} , where $m = 1, 2, \dots$. Typically the first few rows behave smoothly, while higher order rows become erratic. We retain only those lower order rows which behave smoothly. In the example given here in table 2, the first six rows behave smoothly—by which we mean monotonically. This is unusually good behaviour. Frequently rather fewer rows are monotonic. There is a breakdown of monotonicity in the last row. One would estimate from this table that the limiting value was 1.999 9999, and it wouldn't be considered unreasonable to conjecture that the limit is exactly 2.

We will continue the analysis assuming $\sigma = 1/2$ and $\mu = 2$ in order to estimate the other parameters, μ_1 and g in the asymptotic form (13). From (16), one has

$$r_n/2 = 1 + \frac{\log \mu_1}{2\sqrt{n}} + \frac{g + \frac{1}{8} \log^2 \mu_1}{n} + O(n^{-3/2}). \tag{30}$$

In order to estimate μ_1 and g , we solve, sequentially, the pair of equations

$$r_j/2 = 1 + \frac{c_1}{\sqrt{j}} + \frac{c_2}{j}, \tag{31}$$

for $j = k - 1$ and $j = k$, with k ranging from 1 up to 50.

The results are shown in figures 5(a) and (b), giving estimates of the parameters c_1 and c_2 respectively. The first neglected term in (30) is $O(n^{-3/2})$ which is $O(1/n)$ smaller than the term with coefficient c_1 , so c_1 is plotted against $1/n$. By a similar argument, c_2 is plotted against $1/\sqrt{n}$. A simple extrapolation, literally with a straight-edge, gives the estimates $c_1 \approx 1.00$ and $c_2 \approx -0.25$. From (30), $c_1 = \log \mu_1/2$ and $c_2 = g + \log^2 \mu_1/8$. Hence we estimate $\log \mu_1 \approx 2$, and $g \approx -0.75$. As it happens, one sees from equation (29) that these values are exact.

We next tried the idea of direct fitting to the coefficients, as described in section 5.2. Recall that this involves fitting the logarithm of the coefficients to the assumed form and solving successive quartets of equations. Still using just 50 terms in the generating function (28), we estimate $c_1 \approx 0.6932$, implying $\mu \approx 2.0001$, (recall that it is exactly 2), $c_2 = 1.999$, implying $\mu_1 \approx \exp(1.999)$, (recall that it is exactly $\exp(2)$), $c_3 \approx -0.77$, compared to the exact value -0.75 , and $c_4 \approx -1.8$ implying $B \approx 0.16$, compared to the exact value $B = 0.171\ 09\dots$.

These estimates were obtained quite simply by plotting the successive estimates of each parameter against $1/n$ and visually extrapolating. In each case, without wishing to be too precise, we expect errors to be confined to the last quoted digit.

Fitting to three parameters, imposing the fact that $\mu = 2$ is known (or guessing it from the results of the above analysis), the remaining parameters are estimated with significantly improved precision. In this way we estimate $c_2 \approx 1.9995$, $c_3 \approx -0.75$ and $c_4 \approx -1.77$, (the exact value is 1.765 56...).

Next we apply the method of DAs to the transformed series, as described in section 5.4. The approximants are found to be well converged, and we estimate $x_c = 1/\mu \approx 0.499\ 999\ 89$, which differs from the exact value in the seventh decimal place, and $g \approx -0.749$. These are

Table 2. Last seven entries in each row of the table of Bulirsch–Stoer extrapolants with $w = 1/2$. Each successive row is the result of a successively higher degree of extrapolation. The available number of coefficients for extrapolation is N (50 in this example). The highest order estimates, and presumably most precise, are all in the last column.

L	$T(L, N-L-6)$	$T(L, N-L-5)$	$T(L, N-L-4)$	$T(L, N-L-3)$	$T(L, N-L-2)$	$T(L, N-L-1)$	$T(L, N-L)$
1	2.044 802 68	2.043 899 59	2.043 033 43	2.042 201 94	2.041 403 04	2.040 634 83	2.039 895 54
2	1.996 766 53	1.996 888 12	1.997 002 39	1.997 109 94	1.997 211 30	1.997 306 95	1.997 397 34
3	2.000 116 50	2.000 111 19	2.000 106 22	2.000 101 56	2.000 097 19	2.000 093 09	2.000 089 24
4	2.000 004 01	2.000 003 47	2.000 002 99	2.000 002 58	2.000 002 22	2.000 001 90	2.000 001 63
5	1.999 992 80	1.999 993 60	1.999 994 29	1.999 994 88	1.999 995 39	1.999 995 83	1.999 996 22
6	1.999 998 62	1.999 998 70	1.999 998 78	1.999 998 85	1.999 998 91	1.999 998 97	1.999 999 02
7	1.999 999 91	1.999 999 90	1.999 999 89	1.999 999 89	1.999 999 89	1.999 999 90	1.999 999 90

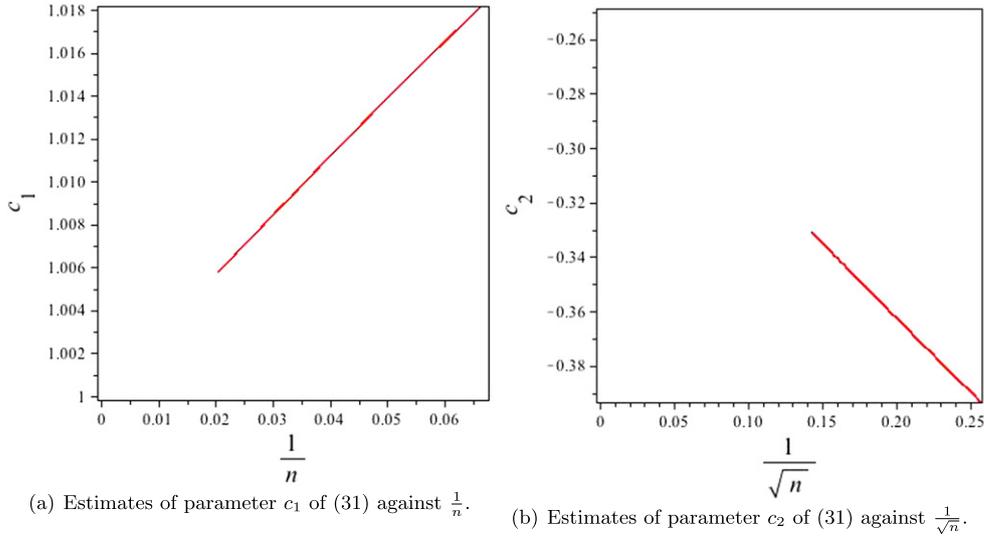


Figure 5. Estimates of parameters c_1 and c_2 from equation (31) for fragmented permutations.

quite close to the exact values $1/2$ and $-3/4$ respectively. However, this is not a particularly testing example, as the original generating function is holonomic.

We also extrapolated the ratios of the coefficients of the transformed series using the Bulirsch–Stoer algorithm, with parameter $w = 1$, as appropriate for an expected correction term $O(1/n)$, which arises when taking ratios. We estimate $\mu \approx 2.000\,000\,60$, compared to the exact value 2.0 .

We can also use the transformed series to directly estimate the exponent g , either imposing prior knowledge of the growth constant μ or not. In this instance we did not assume the value of μ was known, and so extrapolated estimators $g_n = n^2(1 - \frac{r_n}{r_{n-1}})$ of the exponent g against $1/n$, as described by equation (5). In this way we estimated $g \approx -0.755$, compared to the exact value $-3/4$.

Finally, to estimate the amplitude, we did the most obvious thing and divided the coefficient f_n by the terms we’ve already identified in the asymptotic form of the coefficients. That is, we calculated the sequence

$$B_n = \frac{f_n \cdot n^{3/4}}{2^n \cdot \mu_1^{\sqrt{n}}}$$

with $\mu_1 = \exp(2)$. Extrapolating the first 50 values B_n against $1/n^{3/4}$ gave a straight line which could be extrapolated, just with a straight-edge, to give the estimate $B \approx 0.1705$. The exact value is $B = 0.171\,099\dots$

So for this rather simple example we see that the suite of methods we have developed combine to give good numerical estimates of the critical parameters in the asymptotic form of the coefficients. We emphasize that the estimates are predicated on correctly identifying the exponent σ .

7. Example 2. Dyck paths enumerated by maximum height

As the second example, we consider the problem of Dyck paths enumerated not just by length, but also by height, which we define to be the maximum vertical distance of a Dyck path from the horizontal axis. Let $d_{n,h}$ be the number of Dyck paths of length $2n$ and height h , so the OGF is

$$D(x, y) = \sum_{n,h} d_{n,h} x^{2n} y^h.$$

Then

$$\left[x^{2n} \right] D(x, y) = \sum_{h=1}^n d_{n,h} y^h. \tag{32}$$

For $y > 1$,

$$D(x, y) \sim \frac{\text{const.}}{x_c(y)^2 - x^2},$$

where $x_c(y) = \frac{y}{(y+1)}$, and the constant is y -dependent. For $y = 1$, the well-known result is

$$D(x, 1) = \frac{1 - \sqrt{1 - 4x^2}}{2x^2},$$

and for $y < 1$ the solution is usually given as an infinite sum of algebraic functions, from which the asymptotic behaviour is difficult to extract. However, it is possible to do so [24]¹¹, and with constants $A = 2^{5/3} \pi^{5/6} / \sqrt{3}$, $C = 3 \left(\frac{\pi}{2} \right)^{2/3}$ and $r = -\log y$, this is

$$\left[x^{2n} \right] D(x, y) = \frac{(1-y)}{y^2} r^{1/3} A 4^n n^{-5/6} e^{-Cr^{2/3} n^{1/3}} \left(1 + O\left(n^{-1/3} \right) \right). \tag{33}$$

So for $y < 1$ we see that coefficients of Dyck paths, indexed by length and height, behave as (13), with $B = \frac{(1-y)}{y^2} r^{1/3} A$, $\mu = 4$, $\mu_1 = \exp(-Cr^{2/3})$, $\sigma = \frac{1}{3}$, and $g = -\frac{5}{6}$.

In the next subsection we will attempt to determine the critical parameters, assuming the coefficients have the generic asymptotic behaviour (3) from the analysis of the series, with y chosen to be 0.5, and using just 50 terms in the series (we actually generated 2500).

7.1. Numerical analysis of Dyck path series

The series is an even function, so we make the substitution $v = x^2$. So $x_c = 1/2$ becomes $v_c = 1/4$. Applying the method of DAs to the original series, the (very poorly converged) approximants suggest the presence of a singularity at $v_c \approx 0.2511$ (rather than 0.250 00), and with critical exponent in the range [6, 8], which is both an unlikely value and a very imprecise one. Furthermore, the approximants suggest that there are other singularities on the real axis at $v \approx 0.256$, (with an exponent around 15), $v \approx 0.286$, (with an exponent around -12), $v \approx 0.328$, (with an exponent around -1.7) and poles at $v \approx 0.381$, $v \approx 0.500$, $v \approx 0.643$, and $v \approx 1$ ¹². So this is our first indication that the singularity is non-algebraic. Accordingly,

¹¹ I posed this problem at an Oberwolfach meeting in March 2014. Within hours Robin Pemantle confirmed the subdominant behaviour $\mu_1^{n^{1/3}}$, and within 24 h Brendan McKay gave the complete solution of the dominant asymptotic behaviour given above. Subsequently Nick Beaton derived the sub-dominant term.

¹² The exact solution, in terms ratios of Chebyshev polynomials of the second kind, immediately implies the presence of singularities along the ray $v > 1/4$ and which are dense on that ray.

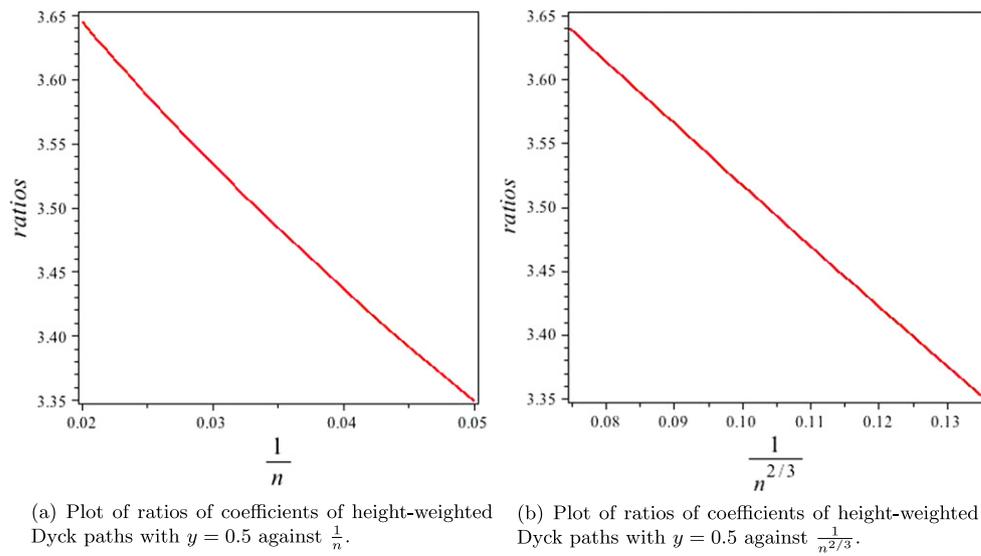


Figure 6. Ratio plots for pushed Dyck paths.

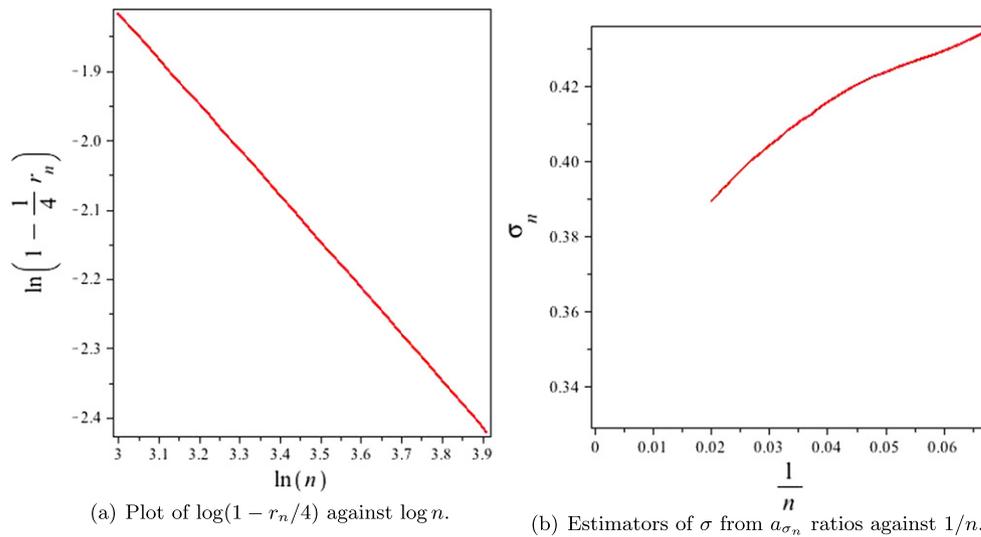


Figure 7. Plots to determine the exponent σ for Dyck path asymptotics.

we test for the plausibility that the appropriate asymptotic form of the coefficients are given by equation (13).

We assume that we don't know the asymptotic form (33), but just have the first 50 terms in the expansion, for $y = 0.5$. We repeat the analysis used in the previous example. We first try a simple ratio plot, the result of which is shown in figure 6(a). Some curvature is evident, though not as much as in figure 3(a), which is not surprising as from (17) we expect the ratios to become linear when plotted against $1/n^{2/3}$ whereas in the case of fragmented permutations, the appropriate abscissa was $1/\sqrt{n}$. In figure 6(b) we show the ratios plotted against $1/n^{2/3}$,

which looks visually linear, and also to be approaching the expected limit of 4. Plotting the ratios against $1/\sqrt{n}$ looks almost as linear, and similarly to the previous example, this crude linearity test only allows us to estimate the value of σ to be in the range [0.4, 0.7].

In order to better estimate the value of σ we show in figure 7(a) a plot of $\log(1 - r_n/\mu)$ against $\log n$. This is seen to be linear, and the gradient, calculated from the last two ratios, r_{49} and r_{50} is -0.675 . If one accepts that σ is likely to be a simple rational number, the value of $1/3$ is the most compelling guess. (Recall that the gradient of this plot should be $\sigma - 1$.)¹³ Estimating σ from the second estimator (20) gives a less compelling result, as the gradient, calculated from the last two ratios, gives $\sigma \approx 0.41$. However a plot of these gradients against $1/n^{1/3}$ is clearly approaching a limiting value around 0.33.

Alternatively, if we didn't know the value of μ , we could estimate σ from the gradient of plots of $\log(r_{\sigma n})$ against $\log n$, see equation (22), or $\log(a_{\sigma n})$, see equation (23). It turns out that estimators from (22) are decreasing below $1/3$, only turning around after some 300 terms. However estimators from (23) are quite informative, and we show in figure 7(b) the estimate of σ given by the extrapolated gradients as n ranges from 15 to 50. There is some curvature in the plot, but clearly a limit of $1/3$ is quite plausible.

Again we see that if one wants to identify σ as a simple fraction, likely to be $1/2$ or $1/3$, then we have good evidence that it is $1/3$. However we couldn't distinguish this fraction from, say, $13/40 = 0.325$.

In the subsequent analysis we assume that $\sigma = 1/3$. We next require a good estimate of μ , which from ratio plots we know to be around 4. As in the preceding example, we can extrapolate the ratios using the Bulirsch–Stoer algorithm, this time with parameter $w = 2/3$. The results are given in table 3. The first four rows behave smoothly—by which we mean monotonically. The monotonicity breaks down in the fifth row. One would estimate from this table that the limiting value was around 4.001, and one might conjecture that the limit is exactly 4. If one uses 100 terms instead of 50, the last entries are around 4.0008 and slowly declining. For the remainder of the analysis we assume $\mu = 4.000$. (Not much changes if we use 4.001.)

In order to estimate μ_1 and g , recall that from (17), it follows that

$$r_n/4 = 1 + \frac{\log \mu_1}{3n^{2/3}} + \frac{g}{n} + O(n^{-4/3}). \tag{34}$$

As in the previous example, we solve, sequentially, the pair of equations

$$r_j/4 = 1 + \frac{c_1}{j^{2/3}} + \frac{c_2}{j}, \tag{35}$$

for $j = k - 1$ and $j = k$, with k ranging from 1 up to 50.

The results are shown in figures 8(a) and (b), the ordinates giving estimates of the parameters c_1 and c_2 respectively. The first neglected term in equation (34) is $O(n^{-4/3})$ which is $O(1/n^{2/3})$ smaller than the term with coefficient c_1 , so c_1 is plotted against $1/n^{2/3}$. By a similar argument, c_2 is plotted against $1/n^{1/3}$. A simple extrapolation, literally with a straight-edge, gives the estimate $c_1 \approx -1.05$. The plot for c_2 exhibits some curvature, and the best we can estimate is $c_2 \approx -1$. From (17), $c_1 = \log \mu_1/3$ and $c_2 = g$. Hence we estimate $\log \mu_1 \approx -3.15$, and $g \approx -1$. The exact values are $\log \mu_1 = -3.175\dots$ and $g = -5/6$. If we take 100 terms in the expansion instead of the 50 that we've used, this method gives the more

¹³ A more detailed analysis can be conducted, in which the estimates of the gradient formed from increasing successive pairs of ratios, r_n and r_{n-1} are extrapolated against $1/n^{1/3}$, and this does indeed give a value around 0.667, but we don't consider that refinement necessary for this example.

Table 3. Last seven entries in each row of the table of Bulirsch–Stoer extrapolants. Each successive row is the result of a successively higher degree of extrapolation. The available number of coefficients for extrapolation is N (50 in this example). The highest order estimates, and presumably most precise, are all in the last column, (apart from the last entry).

L	$T(L, N-L-6)$	$T(L, N-L-5)$	$T(L, N-L-4)$	$T(L, N-L-3)$	$T(L, N-L-2)$	$T(L, N-L-1)$	$T(L, N-L)$
1	4.051 666 25	4.050 405 92	4.049 204 22	4.048 057 41	4.046 961 96	4.045 914 57	4.044 912 14
2	4.013 395 30	4.013 053 70	4.012 737 85	4.012 442 88	4.012 164 55	4.011 899 44	4.011 644 88
3	4.003 674 31	4.004 018 02	4.004 251 09	4.004 376 41	4.004 405 15	4.004 353 26	4.004 238 94
4	4.007 624 63	4.007 354 17	4.006 821 91	4.006 004 06	4.004 857 67	4.003 343 52	4.001 476 59
5	4.007 614 75	4.005 676 28	4.004 952 34	4.004 595 79	4.004 423 33	4.004 382 06	4.004 471 12

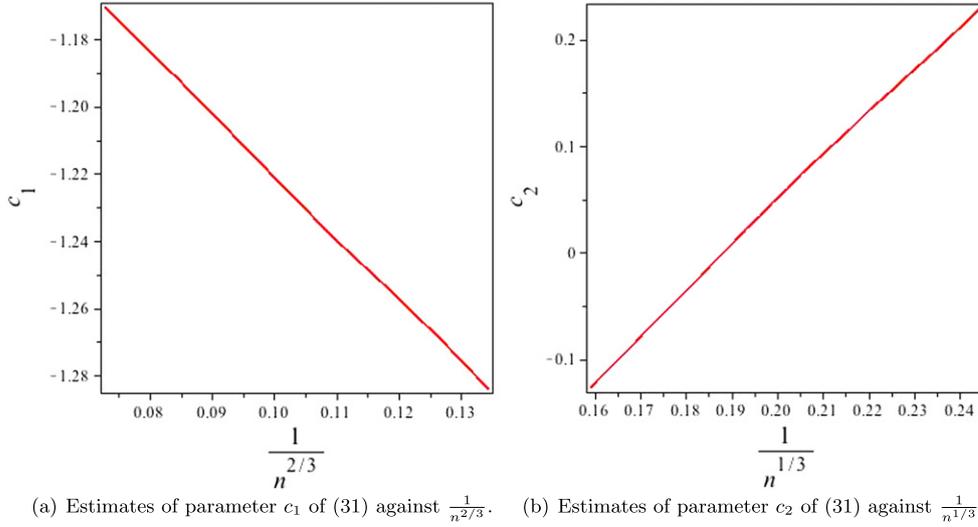


Figure 8. Estimates of parameters c_1 and c_2 from equation (31) for pushed Dyck paths.

accurate results $\log \mu_1 = -3.171\dots$ and $g = -0.83$. As we expect critical exponents to be simple rational fractions, the exact value $g = -5/6$ may well be guessed.

Assuming $\sigma = 1/3$ and directly fitting to the remaining parameters, as described in section 5.2 above, we estimate $c_1 \approx 1.3868$, implying $\mu \approx 4.002$ rather than the exact value of 4, $c_2 \approx -3.28$ rather than the exact value $-3.175\dots$, c_3 is in the range $[-0.9, -0.7]$, compared to the exact value of $-5/6$, and $c_4 \approx 1.8$ rather than the exact value of 2.1308...

If, in addition, we assume that $\mu = 4$ and fit to the remaining three parameters, we find $c_2 \approx -3.20$ rather than the exact value $-3.175\dots$, $c_3 \approx -0.78$, compared to the exact value of $-5/6$, and $c_4 \approx 1.95$ rather than the exact value of 2.1308...

We next considered the transformed series (27). The DAs, applied to the transformed series, while useful, are not as well converged as those in the previous example, with estimates of x_c differing from the exact value in the fifth decimal place, allowing the useful estimate $x_c \approx 0.249\ 98$. The corresponding exponent estimate is $g \approx -0.80$, which can be compared to the correct value $-0.833\ 33\dots$.

We extrapolated the ratios of the coefficients of the transformed series using the Bulirsch–Stoer algorithm, with parameter $w = 1$. We estimate $\mu \leq 4.000\ 36$, compared to the exact value of 4.0. This is more precise than the same analysis applied to the original series.

As in the previous example, we estimated the exponent g by extrapolating estimators $g_n = n^2(1 - \frac{r_n}{r_{n-1}})$ against $1/n$, as described at equation (5). For this example we estimate $g \approx -0.84$, compared to the exact value of $-5/6$.

Finally, to estimate the amplitude, we did as with the first example and divided the coefficients by the terms we’ve identified in the asymptotic form of the coefficients. That is, we calculated the sequence

$$B_n = \frac{[x^{2n}]D(x, y) \cdot n^{5/6}}{4^n \cdot \mu_1^{n^{1/3}}}$$

with μ_1 taken to be in the range $[0.0405, 0.043]$, from the various estimates found above. The exact value is $\mu_1 = 0.041\ 79\dots$. Extrapolating the first 50 values B_n against $1/n^{3/4}$ gave a

straight line which could be extrapolated, just with a straight-edge, to give an estimate of B in the rather broad range [7.3, 9.6]. This large variation is due entirely to the uncertainty in the value of μ_1 . Using the correct value of μ_1 leads to the estimate $B \approx 8.4$. The exact value is $B = 8.422\ 08\dots$.

This example displays behaviour typical of that which we have encountered in other problems, such as SAWs, SAPs and bridges subject to a compressive force. It can be seen that the methods we have developed can clearly identify the nature of the singularity, and also provide good estimates of the various critical parameters, provided a sufficient number of coefficients is known.

8. Example 3. IPDSAWs

For our third and final example, we consider IPDSAW. These are random walks on the square lattice with both west steps and immediate reversals forbidden. The two constraints immediately imply that the paths are self-avoiding. Paths are counted by length, and by the number of monomer–monomer interactions, which occur between adjacent sites that are not consecutive vertices of the walk. The appropriate OGF is

$$G(x, y) = \sum_{n,h} c_{n,m} x^n y^m,$$

where $c_{n,m}$ is the number of n -step IPDSAWs with m monomer–monomer interactions. Then

$$[x^n]G(x, y) = \sum_{m=1}^n c_{n,m} y^m. \tag{36}$$

This model was solved in [3]. Let

$$g_0 = 1 + \sum_{j=1}^{\infty} \frac{x^{2j} (x - q)^j q^{j(j+1)/2}}{\prod_{i=1}^j (xq^i - x)(xq^i - q)}$$

and

$$g_1 = x + x \sum_{j=1}^{\infty} \frac{x^{2j} (x - q)^j q^{j(j+1)/2} q^j}{\prod_{i=1}^j (xq^i - x)(xq^i - q)},$$

where $q = xy$. Then for $y \neq 1$, with $a = x^2(2 - 4x)$ and $b = x^2(6 - 4x)$, the solution is

$$G(x, y) = \frac{2xg_1 - ag_0}{bg_0 - 2xg_1}. \tag{37}$$

The asymptotic form of the coefficients is difficult to extract from (37), but based on an analysis of a 6000 term series, Owczarek *et al* [26] conjectured the asymptotic form numerically as $B \cdot \mu^n \cdot \mu_1^{\sqrt{n}} \cdot n^{-3/4}$, where both $\mu > 1$ and $\mu_1 < 1$ depend on the monomer–monomer interaction strength y , in the collapsed regime $y > y_c \approx 3.383$.

Recently, Nguyen and Pétrélis [24] have given a more probabilistic exposition of this problem, which has the advantage that the term $\mu_1^{\sqrt{n}}$ in the asymptotic form of the coefficients in the collapsed regime is seen as a natural consequence of the law governing a symmetric random walk. As an aside, we remark that Pemantle’s argument (footnote 9) for a term of the form $\mu_1^{n^{1/3}}$ arising in the Dyck path case just discussed is a consequence of the law for reflected Brownian bridges.

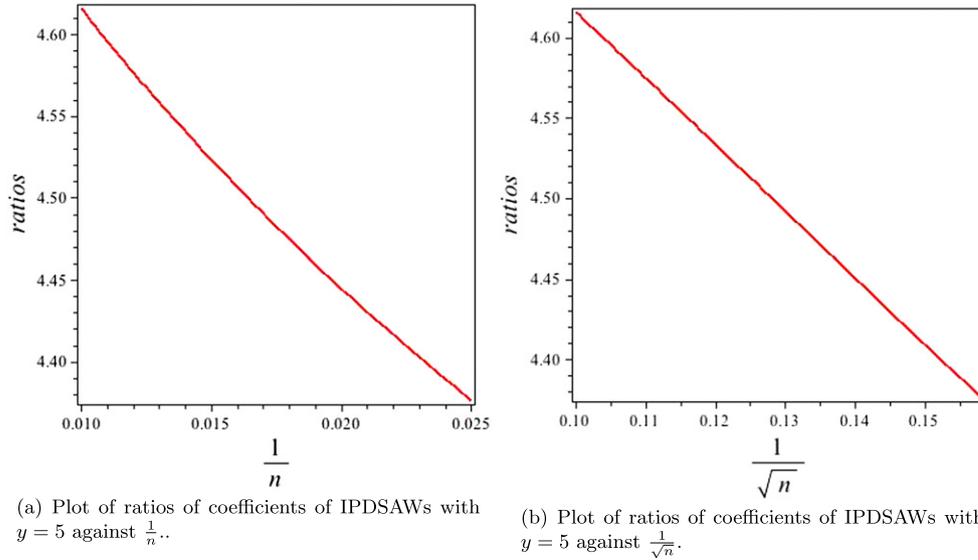


Figure 9. Ratio plots for IPDSAWs.

We expanded (37) to obtain 100 terms in the series. It was necessary to obtain somewhat longer series than in our previous examples, as the low order terms involve no monomer–monomer interactions, and it is not until about length 20 that a significant number of interactions occur. As shown in [3], the tricritical point occurs at $(x_c, y_c) = (1/y_c, y_c)$, where $y_c \approx 3.382\,975\dots$. For $y > y_c$ there is a line of critical points lying on the hyperbola $xy = 1$. As long as we choose a value of $y > y_c$, we are in the so-called collapsed regime, where the coefficients have the asymptotic form (13). For simplicity we have chosen $y = 5$, so the generating function $G(x, 5)$ will have a critical point at $x_c = 1/5$, so $\mu = 5$ in equation (13).

In this example, constructing DAs to the original series gives very poorly converged results, which as discussed above is an indication that the underlying OGF does not have a dominant algebraic singularity. The approximants suggest that the critical point is around 0.205 (rather than 0.2 exactly), with an exponent in the range [5, 8]. Again, this large numerical value for the exponent, and its imprecision, suggests that a non-algebraic singularity is dominant.

As before, we first plot the ratios of successive terms against $1/n$, as shown in figure 9(a). Some curvature is evident. We next plot the same ratios in figure 9(b) against $1/\sqrt{n}$, and the plot is seen to be visually linear, implying $\sigma \approx 1/2$.

To more accurately determine the value of σ , we show in figure 10(a) a plot of $\log(1 - r_n/\mu)$ against $\log n$. This is seen to be linear, and the gradient, calculated from the last two ratios r_{99} and r_{100} is -0.532 . If one accepts that σ is likely to be a simple rational number, the value $1/2$ is inescapable¹⁴. Estimating σ from the second estimator (20) gives a similar result, as the gradient, calculated from the last two ratios, gives $\sigma \approx 0.474$.

Again, if we didn't know the value of μ , we could estimate σ from the gradient of plots of $\log(r_{\sigma_n})$ against $\log n$, see equation (22), or $\log(a_{\sigma_n})$, see equation (23). It turns out, in contrast to the situation with the previous example, that estimators from (23) require hundreds of terms before a clear approach to the limit can be seen. However estimators from (22) are

¹⁴ With 200 terms the gradient estimate is improved to -0.522 .

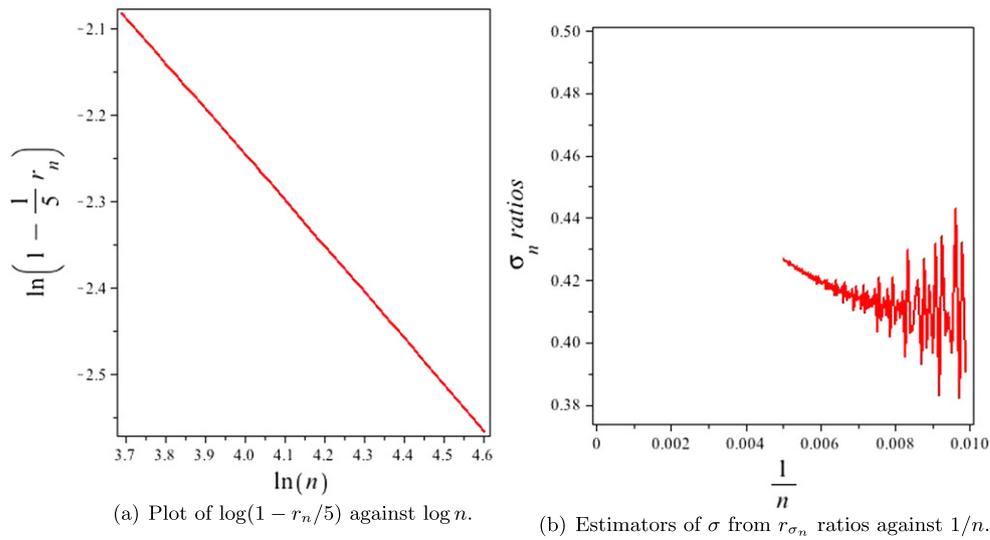


Figure 10. Plots to determine the exponent σ for IPDSAW asymptotics.

quite informative, though we still require 200 terms to draw any convincing conclusions. We show in figure 10(b) the estimate of σ given by the gradient of the line joining the points r_{σ_k} and $r_{\sigma_{k-1}}$, as k ranges from 100 to 200. There is some curvature and oscillation in the plot, but clearly a limit of $1/2$ is attainable.

Once again we see that if one wants to identify σ as a simple fraction, likely to be $1/2$ or $1/3$, then we have good evidence that it is $1/2$. This can then be used in subsequent analysis.

As in the preceding examples, we can also extrapolate the ratios using the Bulirsch–Stoer algorithm, this time with parameter $w = 1/2$. The results are shown in table 4. Only the first row behaves smoothly—by which we mean monotonically. The monotonicity breaks down already in the second row. This is not totally surprising, as this series behaves slightly erratically, like the number of partitions of the integers. The number of interactions is not a fixed fraction of the length, and so low-order ratio plots are a little erratic going from one term to the next, though the global trend is uniform. One might estimate from this table that the limiting value was around 5.00, and a brave person might conjecture that the limit is exactly 5. If one uses 200 terms instead of 100, the last entries are around 4.9995.

Assuming then that $\sigma = 1/2$, and $\mu = 5$, from (16) it follows that

$$r_n/5 = 1 + \frac{\log \mu_1}{2\sqrt{n}} + \frac{g + \frac{1}{8} \log^2 \mu_1}{n} + O(n^{-3/2}).$$

As in example 1, in order to estimate μ_1 and g , we solve, sequentially, the pair of equations

$$r_j/5 = 1 + \frac{c_1}{\sqrt{j}} + \frac{c_2}{j}, \tag{38}$$

for $j = k - 1$ and $j = k$, with k ranging from 2 up to 200.

The results are shown in figures 11(a) and (b). Simple visual extrapolation using the data points up to $n = 100$ gives the estimate $c_1 \approx -0.71$. The plot for c_2 exhibits some curvature, and the best we can estimate is $c_2 \approx -0.65$ from just 100 terms (note the negative gradient

Table 4. Last seven entries in each row of the table of Bulirsch–Stoer extrapolants. Each successive row is the result of a successively higher degree of extrapolation. The available number of coefficients for extrapolation is N (100 in this example). The highest order estimates, and presumably most precise, are in the last column.

L	$T(L, N-L-6)$	$T(L, N-L-5)$	$T(L, N-L-4)$	$T(L, N-L-3)$	$T(L, N-L-2)$	$T(L, N-L-1)$	$T(L, N-L)$
1	5.068 383 54	5.067 587 79	5.067 120 79	5.066 064 62	5.065 484 13	5.064 808 80	5.064 124 56
2	5.003 262 29	5.003 010 24	5.026 507 00	4.982 912 97	5.015 223 99	5.006 918 75	5.005 103 06

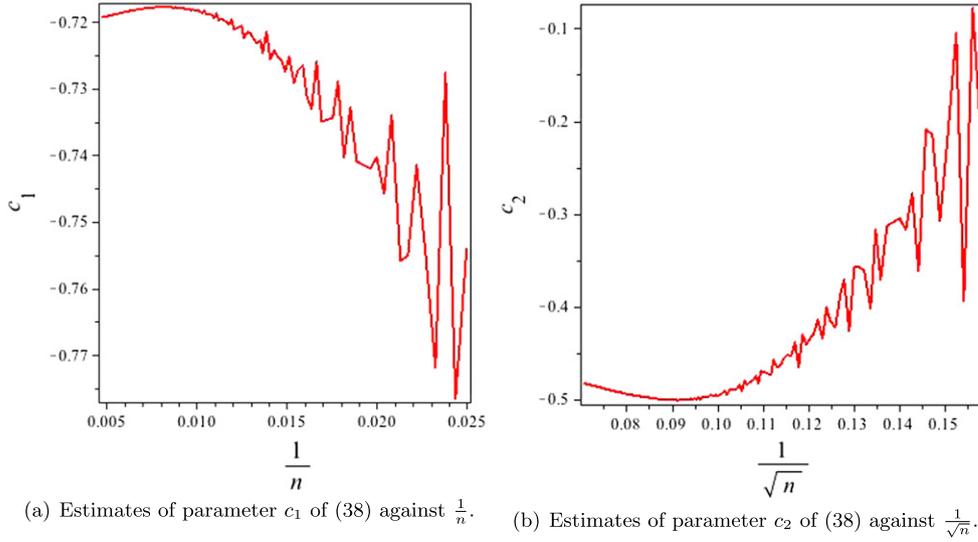


Figure 11. Estimates of parameters c_1 and c_2 from equation (38) for IPDSAWs.

when $1/\sqrt{n} > 0.1$). From (16) these estimates imply $\log \mu_1 \approx -1.42$, and $g \approx -0.9$. If we take 200 terms in the expansion instead of 100, we see that the plots have turning points at around $n = 100$, and that with $n = 200$ our estimate of c_2 is close to -0.4 . The 200 term series lets us make the more precise estimates $\log \mu_1 \approx -1.44$ and $g \approx -0.66$. It turns out that we need some 500 terms in the series before we can confidently estimate $g \approx -0.750$. We also estimated $\log \mu_1 \approx -1.4396$ from a 500 terms series. This agrees with the analysis in [26] based on a 6000 term series, though they claim a more accurate estimate of μ_1 , which is not given.

For this example the direct fitting method was somewhat less successful, as there is a substantial degree of oscillation in the plots of the various parameters, due to parity effects, as discussed above. Nevertheless, the results were useful, and if we use more than 100 terms, quite good accuracy can be achieved. But just using 100 terms, assuming $\sigma = 1/2$ and directly fitting to the remaining parameters, as described in section 5.2 above, we estimate $c_1 \approx 1.61$, implying $\mu \approx 5.003$ rather than the exact value of 5, $c_2 \approx -1.5$ so $\log(\mu_1) \approx -1.5$ rather than the more precise value -1.439 , c_3 is in the range $[-1.4, -0.5]$, compared to the exact value of $-3/4$, and $c_4 \approx 4$ rather than the actual value of around 0.8.

If we assume that $\mu = 5$ and fit to the remaining three parameters, we estimate $c_2 \approx -1.415$ so $\log(\mu_1) \approx -1.415$ rather than the more precise value -1.439 , $c_3 \approx -0.6$, compared to the exact value of $-3/4$, while $c_4 \approx 1.4$ is still a rather poor estimate. If we use 200 terms, the estimates improve to $c_2 \approx -1.44$, $c_3 \approx -0.75$, and $c_4 \approx 0.9$.

For IPDSAWs, as discussed above, the series do not behave smoothly at low order, due to the rather granular way the number of interactions increases with the length of the walk. So even the transformed series are not well-suited to analysis by the method of DAs. Nevertheless, the results of this approach are not without value. The critical point is estimated to be at $x_c \approx 0.2016$, but with a second singularity very close by at $x \approx 0.208$. The two singularities have associated exponents of opposite sign and varying magnitude, so that g cannot be estimated this way.

We extrapolated the ratios of the coefficients of the transformed series using the Bulirsch–Stoer algorithm, with parameter $w = 1$. For IPDSAW the Bulirsch–Stoer extrapolants are, as expected, not monotonic, but do quickly settle down to values in the range [4.9996, 5.0010], in reasonable agreement with the exact value of 5.0.

In order to estimate the critical exponent g , we extrapolated estimators $g_n = n^2(1 - \frac{r_n}{r_{n-1}})$ of the exponent against $1/n$, as described at equation (5). The extrapolants are not monotonic, but do quickly settle down to values in the range $[-1, -0.7]$. Using 250 terms in the series allows the much more precise estimate $g \approx -0.78$, and with 500 terms that improves to $g = -0.75$. The exact value is $g = -3/4$. Note that these values were obtained without recourse to knowledge of the value of the growth constant μ .

As with the previous two examples, we estimated the amplitude by dividing the coefficients by the terms we've identified in the asymptotic form. The uncertainty in the value of μ_1 again gives rise to a rather large uncertainty in the estimate of the amplitude B . With the correct value of μ_1 (or, rather, correct to four significant digits), this procedure gave $B \approx 2.22$ with a 100 term series. The estimate was slightly improved to $B \approx 2.210$ with a 700 term series.

This example represents the most difficult problem of this class, one in which the coefficients do not vary smoothly, and yet one has a singularity of the non-algebraic type that we are studying here. Despite this, a clear indication of the nature of the singularity was obtained, and reasonably accurate estimates of the critical parameters were also obtained, provided one has sufficient series coefficients.

9. General methods to analyse such series

On the basis of these three examples, and others we have studied but not discussed at length here, we are now in a position to propose a method for analysing problems that may have coefficients of the assumed asymptotic form (3).

- Make a plot of the ratios against $1/n$. If this plot is linear, or approaching linearity as n increases, this is suggestive of an algebraic singularity.
- Analyse the series by the method of DAs. If one obtains well-converged estimates of the position of the critical point(s) and exponent(s), and these are consistent with the ratio analysis, this is further evidence for an algebraic singularity. In those cases when the convergence is rapid and precise, as in the example of hexagonal SAPs in section 4.1, one can have abundant confidence in this conclusion.
- If however the DAs are not well converged, and the associated exponent is poorly estimated and considered unlikely for a problem of the class being studied, then there is good reason to doubt that the underlying singularity is algebraic.
- If the ratio plot can be made linear by plotting the ratios against $1/\sqrt{n}$ or $1/n^{2/3}$, or other simple rational exponent $1/n^{1-\sigma}$, this is then further evidence suggesting that the asymptotic form is not algebraic, but rather of the form (13). If one knows, or can accurately estimate, the radius of convergence x_c , then denoting the ratio of successive coefficients by r_n , a plot of $\log |r_n x_c - 1|$ against $\log n$ should give an estimate of the exponent needed to linearize the ratio plot. Alternatively, or additionally, an estimate of σ can be found from a plot of $\log(c_n)$, as defined by equation (19), against $\log n$. This should be consistent with that found by choosing an exponent by trial and error to linearize the ratio plot. Otherwise, if μ is unknown, σ can be estimated from the gradient

of plots of $\log(r_{\sigma_n})$ and/or $\log(a_{\sigma_n})$ against $\log n$, where these quantities are defined in equations (22) and (23) respectively.

- The Bulirsch–Stoer or other appropriate extrapolation algorithm should be used to extrapolate the ratios of the coefficients in order to get a more reliable estimate of the growth constant μ . If the Bulirsch–Stoer algorithm is used, the parameter $w = 1 - \sigma$, where σ is estimated from the value needed to linearize the ratio plots. The estimate of μ can then be used in a log–log plot to refine the estimate of σ , using the estimators defined in equations (18) and (20). In this way a consistent pair of estimates of both μ and σ can be obtained.
- Once the exponent σ is well established, a four parameter fit to the assumed asymptotic form, as described in section 5.2 should be conducted. Using the best estimate, or exact knowledge, of the critical point, $1/\mu$, this should be incorporated, and a three parameter fit tried. Naturally, parameter estimates by different methods should be mutually consistent.
- Then an analysis which involves fitting successive pairs of ratios to $r_n \cdot x_c = 1 + \frac{c_1}{n^{1-\sigma}} + \frac{c_2}{n}$ and extrapolating estimates of c_1 and c_2 should be performed. The estimates of c_1 and c_2 provide estimators of μ_1 and g . If one has very many terms, this can be taken further, and successive triples of ratios can be used to estimate the coefficients c_i , $i = 1, 2, 3$ in $r_n \cdot x_c = 1 + \frac{c_1}{n^{1-\sigma}} + \frac{c_2}{n} + \frac{c_3}{n^{2-2\sigma}}$.
- Finally, the series should be transformed to remove the factor $\mu_1^{n^\sigma}$, as discussed in section 5.4, and the transformed series subject to standard analysis, such as ratio analysis, DA analysis and ratio extrapolation by the Bulirsch–Stoer algorithm. The estimates of the critical parameters should be consistent with, and hopefully more precise than, those obtained by the other methods described.

10. Conclusion

We have described a number of methods to distinguish between coefficients of a generating function with algebraic singularities and those with more complicated non-algebraic singularities with asymptotic behaviour of the form $B \cdot \mu^n \cdot \mu_1^{n^\sigma} \cdot n^g$. We have developed methods to identify coefficients in this latter class, and then shown how to extract estimates for the critical parameters B , μ , μ_1 , σ , and g .

Our methods are based on extending the existing traditional methods for analysing algebraic singularities, the ratio method and the method of DAs. In the latter case we don't extend the method so much as its application. The fairly natural idea of directly fitting to the asymptotic form is also investigated, and found to be useful.

We illustrate these methods by applying them to three examples. They are the generating functions of fragmented permutations, of compressed Dyck paths, and of IPDSAWs. In subsequent papers, of which [7] is the first, we apply these ideas to previously unsolved problems.

Acknowledgments

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asymptotic behaviour of coefficients in the generating function for compressed Dyck paths, as discussed in section 7. I would also like to thank the Australian Research Council who have supported this work through grant DP120100939.

Appendix. Maple implemenation of the Bulirsch–Stoer algorithm

```

# Assume  $s_n \sim s_\infty + c \cdot n^{-w}$  where  $w$  is known or conjectured.
w := 1/2;
# Assume  $s_k$  is known for  $k = 1 \dots n$ . Assume  $n = N_{\max}$ .
n := N_max;
for i from 1 to n do
T[0, i] := 0.0;
T[1, i] := s_i;
end do;
for j from 2 to n do
for i from 1 to n + 1 - j do
tt := T[j - 1, i + 1];
T[j, i] := evalf (tt + (tt - T[j - 1, i])/((i + j)/(i))^w * (T[j - 1, i]
- T[j - 2, i + 1])/(tt - T[j - 2, i + 1]) - 1));
end do;
end do;
for j from 2 to n do
for i from 1 to n + 1 - j do
print(T[j, i]);
end do;
print(j);
end do;

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